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Smooth involutions and splitting invariants revisited

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Abstract

Classification results about smooth involutions of Browder [Trans. Amer. Math. Soc. 178 (1973) 193–225] have been strengthened and as applications smooth splitting invariants along product of real projective spaces and Milnor manifolds embedded in a real projective space have been calculated. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In order to study classification of smooth fixed point free involutions on smooth homotopy spheres (which is same as diffeomorphism classification of smooth homotopy real projective spaces) Browder (cf. [2]) had set up a machinery of generalized version of Kervaire's invariant of even dimensional (mod 2) Poincaré duality spaces. A crucial ingredient of his approach was the following result:

Theorem 1.1 (Motivating result). *For a $2q$ dimensional surgery problem $(f, b) : M'^{2q} \rightarrow M^{2q}$, if q is odd and M^{2q} is 1-connected or if M^{2q} is nonorientable and $\pi_1(M) = \mathbb{Z}/2$, then the surgery obstruction $\sigma(f, b)$ is given by this generalized version of Kervaire's invariant $k(M')$ of M' .*

Consequently a major thrust of that paper was on the computation of this generalized version of the Kervaire's invariant. He proved sum theorems of this invariant under some

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restrictions (see 2.1, 2.3 below), and went on to do certain constructions and computations required for that work. We will recall this very briefly below, because our work will be motivated by this:

Let N^m be a smooth (or p.l., or top.) m -manifold, and $T: N \rightarrow N$ a smooth (or p.l., or top.) fixed point free involution. Let $M^m = N/T$ be the quotient manifold, ξ^k be the normal bundle of M^m in S^{m+k} , k sufficiently large, and let (f, b) be a normal map into M . That is $f: M' \rightarrow M$ is a deg 1 map which is covered by a bundle map of normal bundles $b: \nu_{M'}^k \rightarrow \xi^k$. For $n = 2s + 1$ one defines a new normal map $n(f, b) = (g, c)$ into M as follows:

$$g: nM' \cup -sN \rightarrow M$$

is the map whose restriction on each copy of M' is f and whose restriction on each copy of N is the projection $p: N \rightarrow N/T = M$ of the double covering. So

$$\deg g = n \deg f - s \deg p = n - 2s = 1$$

and c is the bundle map covering g whose restriction on each copy of $\nu_{M'}^k$ is b and whose restriction on each copy of τ_N^{-1} is the inverse of the tangent bundle map

$$d = (dp)^{-1}: \tau_N^{-1} \rightarrow \tau_{M'}^{-1}.$$

Under a technical assumption (refer to 2.6) and the condition that the square of the middle dimensional Wu class satisfies $(v_q(M))^2[M] \neq 0$ Browder could calculate the generalized Kervaire's invariant for this example in terms of the generalized Kervaire's invariant of the original surgery problem (see 2.7 below)

This technical Assumption 2.6 was satisfied by $M^m = \mathbb{R}P^m$ and hence Browder could very efficiently use the machinery to get a lot of information about the diffeomorphism classification of smooth homotopy real projective spaces and there by about the classification of smooth involutions of smooth homotopy spheres.

The purpose of this paper is to first prove the result that the technical Assumption 2.6 mentioned above is in fact satisfied for any double cover $p: N \rightarrow M$. This result constitute the Main Technical Theorem 3.1 of this paper. Therefore we get a strengthened version of Browder's Theorem 2.7 as our main theorem:

Theorem 1.2 (Main theorem). *If $n = 2s + 1$,*

(a) *$k(nM' \cup -sN)$ is defined and $k(nM' \cup -sN) = k(nM')$.*

(b) *If in addition $(v_q(M))^2[M] \neq 0$*

$$k(nM' \cup (-s)N) = \begin{cases} k(M') & \text{for } n \equiv \pm 1 \pmod{8}, \\ k(M') + 1 & \text{for } n \equiv \pm 3 \pmod{8}. \end{cases}$$

This theorem enables one to get many more applications than in [2]. We give applications in calculating splitting invariants along product of real projective spaces, and real Milnor manifolds embedded in a single real projective space of dimension $\equiv 1 \pmod{4}$.

We now recall some terminology needed to discuss these applications. Let $f: X \rightarrow M$ be a homotopy equivalence between manifolds (smooth, pl., or topological). Let $M_1 \subseteq M$

be a submanifold. Then we will call (after making f transversal to M_1) $f^{-1}(M_1)$ a *characteristic submanifold* of X .

As our first application we consider the following situation:

M is $\mathbb{R}P^{4q+1}$ for some large value of q . M_1 is $\mathbb{R}P^r \times \mathbb{R}P^s$ such that $r + s + rs < 4q + 1$, $r, s > 1$. There is a standard embedding $\mathbb{R}P^r \times \mathbb{R}P^s \hookrightarrow \mathbb{R}P^{4q+1}$ given in terms of the homogeneous coordinates of real projective spaces by $i([y_0, \dots, y_r], [z_0, \dots, z_s]) = [y_j z_k]$, $j = 0, 1, 2, \dots, r$; $k = 0, 1, 2, \dots, s$ (see [10, pp. 80–81]).

As our second application we consider the following situation:

M is $\mathbb{R}P^{4q+1}$ as earlier and M_1 is the real Milnor manifold, $\mathbb{R}H_{r,s}$, defined as the codimension 1 submanifold of $\mathbb{R}P^r \times \mathbb{R}P^s$ given in terms of the homogeneous coordinates of the real projective spaces as

$$\mathbb{R}H_{r,s} \stackrel{\text{def}}{=} \{([z_0, z_1, \dots, z_r], [w_0, w_1, \dots, w_s]) \mid z_0 \cdot w_0 + z_1 \cdot w_1 + \dots + z_{\min\{r,s\}} \cdot w_{\min\{r,s\}} = 0\},$$

such that $\dim \mathbb{R}H_{r,s} = r + s - 1$ satisfies $r + s + rs < 4q + 1$, $r, s > 2$.

It is clear that if we take X to be M itself and f to be the identity map, then characteristic submanifolds $f^{-1}(i(\mathbb{R}P^r \times \mathbb{R}P^s))$, or $f^{-1}(\mathbb{R}H_{r,s})$ which are of homotopy types of $\mathbb{R}P^r \times \mathbb{R}P^s$ or $\mathbb{R}H_{r,s}$ respectively, obviously exist for every r, s with the given restrictions. To state our theorems about applications we consider maps which are defined below:

Consider the identity map $1_{\mathbb{R}P^{4q+1}} : \mathbb{R}P^{4q+1} \rightarrow \mathbb{R}P^{4q+1}$.

Then for any $n \equiv \pm 3 \pmod{8}$ define a normal map (g, c) , as above, with

$$g : n\mathbb{R}P^{4q+1} \cup ((n-1)/2)S^{4q+1} \rightarrow \mathbb{R}P^{4q+1}$$

and c as defined earlier. Since the Wall surgery obstruction group $L_{4q+1}(\mathbb{Z}[\mathbb{Z}/2]^+) = 0$, (g, c) is normally cobordant to a map (F, \bar{c}) with $F : N' \rightarrow \mathbb{R}P^{4q+1}$ a homotopy equivalence.

Theorem 1.3 (Main application 1). *The manifold N' admits characteristic submanifold of homotopy type of $\mathbb{R}P^r \times \mathbb{R}P^s$ with $r + s + rs < 4q + 1$, $r, s > 1$ if and only if r and s are not both even.*

In other words splitting invariants $\mathbf{s}_{(r,s)}(F, \bar{c})$, $r + s + rs < 4q + 1$, $r, s > 1$ (in the notation of Wall [11]) are given by

$$\mathbf{s}_{(r,s)}(F, \bar{c}) = \begin{cases} 1 & \text{if } r, \text{ and } s \text{ are even,} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.4 (Main application 2). *The manifold N' admits characteristic submanifold of homotopy type of real Milnor manifold $\mathbb{R}H_{r,s}$ of dimension $r + s - 1$ with $r + s + rs < 4q + 1$, $r \geq s > 2$ if and only if r, s are not of the form: r is odd and s is even.*

In other words splitting invariants $\hat{\mathbf{s}}_{(r,s)}(F, \bar{c})$, $r + s + rs < 4q + 1$, $r \geq s > 2$ (in the notation of Wall [11]) are given by

$$\hat{\mathbf{s}}_{(r,s)}(F, \bar{c}) = \begin{cases} 1 & \text{if } r \text{ is odd, and } s \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

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2. Preparatory materials

Consider a degree 1 map $f: M^{2q} \rightarrow M^{2q}$ (respectively, a double cover $p: N^{2q} \rightarrow M^{2q}$). Let ξ^k be a vector bundle over M , and let $b: \nu_{M'}^k \rightarrow \xi^k$ cover f (respectively, $d = (dp)^{-1}: \tau_N^{-1} \rightarrow \tau_M^{-1}$ cover p), here ν and τ denote the relevant normal and tangent bundles, respectively. The pair (f, b) (respectively, the pair (p, d)) define, either by Hirsch immersion theory, general position arguments, and Pontrjagin–Thom construction (as used in Browder [2]), or using Spanier–Whitehead duality (as used in Browder [3], and [1]), a map $\eta_f: \Sigma^k M_+ \rightarrow \Sigma^k M'_+$ (respectively, a map $\eta_p: \Sigma^k M_+ \rightarrow \Sigma^k N_+$), which we shall continue to call the *spectral orientation map* (with a slight abuse of the terminology used in [2]). If we compose this with $\Sigma^k f$ (respectively, with $\Sigma^k p$) then the homotopy class of the composite map

$$h_f = \Sigma^k f \circ \eta_f: \Sigma^k M_+ \xrightarrow{\eta_f} \Sigma^k M'_+ \xrightarrow{\Sigma^k f} \Sigma^k M_+$$

(respectively,

$$h_p = \Sigma^k f \circ \eta_p: \Sigma^k M_+ \xrightarrow{\eta_p} \Sigma^k N_+ \xrightarrow{\Sigma^k p} \Sigma^k M_+)$$

will be called the *spectral cobordism invariant* (with a slight abuse of the terminology used in [2]) of (f, b) (respectively, of (p, d)).

In a similar manner one can define spectral orientation map η_g and spectral cobordism invariant h_g of the normal map $(g, c) = n(f, b)$, $n = 2s + 1$ described in the introduction. Now, we define the generalized Kervaire's invariant (due to Browder) $k(M')$ (respectively, $k(N)$, and $k(nM' \cup -sN)$) as the Arf invariant of the quadratic form defined on the following subspace of $H^q(M'; \mathbb{Z}/2)$:

$$\Psi: [\ker(\eta_f^* \circ \Sigma^k)]^q \rightarrow \mathbb{Z}/2; \quad \Psi(x) \stackrel{\text{def}}{=} [Sq_{(\Sigma^k \phi) \circ \eta_f}^{q+1}(\Sigma^k \iota)][\Sigma^k M_+],$$

where $\phi: M' \rightarrow K(\mathbb{Z}/2, q)$ is the map such that $\phi^*(\iota) = x$, ι being the fundamental class of $K(\mathbb{Z}/2, q)$, and Sq_0^{q+1} is the functional Steenrod square (respectively, Arf invariant of the quadratic form

$$\Psi: [\ker(\eta_p^* \circ \Sigma^k)]^q \rightarrow \mathbb{Z}/2; \quad \Psi(x) \stackrel{\text{def}}{=} [Sq_{(\Sigma^k \phi) \circ \eta_p}^{q+1}(\Sigma^k \iota)][\Sigma^k M_+],$$

and of

$$\Psi: [\ker(\eta_g^* \circ \Sigma^k)]^q \rightarrow \mathbb{Z}/2; \quad \Psi(x) \stackrel{\text{def}}{=} [Sq_{(\Sigma^k \phi) \circ \eta_g}^{q+1}(\Sigma^k \iota)][\Sigma^k M_+].$$

However the generalized Kervaire's invariant $k(M')$, for instance, is not always defined. A necessary and sufficient condition under which it is defined is

$$\Psi \mid f^* H^q(M; \mathbb{Z}/2) \cap (\ker \eta_f^* \circ \Sigma^k)^q = 0.$$

We now state some results of Browder [2] as ready reference for our work:

Theorem 2.1 (Browder [2, (2.13)]). *Suppose the Kervaire invariant is defined for M'_1 and M'_2 , and $h_{f_2}^* = 0$ on $H^q(M; \mathbb{Z}/2)$. Then the Kervaire invariant is defined for $M'_1 + M'_2$ and $k(M'_1 + M'_2) = k(M'_1) + k(M'_2)$.*

Remark 2.2. As spectral orientations and classifying maps for M' and $-M'$ differ by automorphisms, definability conditions and values of the Kervaire invariant for M' and $-M'$ are same.

Theorem 2.3 (Browder [2, (2.14)]). *Suppose $k(M'_1), k(M'_2)$, and $k(M'_1 + M'_2)$ are all defined and suppose $h_{f_1}^* = h_{f_2}^*$ on $H^q(M; \mathbb{Z}/2)$. Then $k(M'_1 + M'_2) = k(M'_1) + k(M'_2)$.*

Proposition 2.4 (Browder [2, (2.16)]). *Let $x \in H^*(M; \mathbb{Z}/2)$. Then $h_f^*(x) = 0$ if and only if all characteristic numbers involving x are zero.*

Theorem 2.5 (Browder [2, (3.8)]). *$k(2M')$ is defined if and only if $(f^*y)^2[M] = 0$ for all $y \in H^q(M; \mathbb{Z}/2)$.*

Browder in [2, (4.6)] considered the following technical assumption:

Assumption 2.6. For the spectral cobordism invariant h_p of N , assume that:

- (a) $h_p^*: H^{q+k}(\Sigma^k M_+; \mathbb{Z}/2) \rightarrow H^{q+k}(\Sigma^k M_+; \mathbb{Z}/2)$ be the zero map;
- (b) $k(N)$ be defined and $k(N) = 0$ (using the quadratic form defined above).

For M, N, M' etc. as above, he proved *under the above Assumption 2.6*, the following:

Theorem 2.7 (Browder [2, (4.8)]).

(a) *If 2.6 is satisfied and n is odd, $k(nM' \cup -sN)$ is defined and $k(nM' \cup -sN) = k(nM')$.*

(b) *If 2.6 is satisfied and the Wu class of M satisfies $(v_q(M))^2[M] \neq 0$, then*

$$k(nM' \cup -sN) = \begin{cases} k(M') & \text{for } n \equiv \pm 1 \pmod{8}, \\ k(M') + 1 & \text{for } n \equiv \pm 3 \pmod{8}. \end{cases}$$

The above condition 2.6 and hence the above Theorem 2.7 are valid if we take $M = \mathbb{R}P^{2q}, N = S^{2q}, (f, b)$ any normal map of M' into M , and p the double cover.

We want to investigate these results and their consequences when M is (i) a product of real projective spaces, (ii) a real Milnor manifold with N as any double cover over it.

3. The main technical theorem and immediate consequences

Consider a double cover $p: N^{2q} \rightarrow M^{2q}$. Let

$$\eta_p: \Sigma^k M_+ \rightarrow \Sigma^k N_+$$

be the spectral orientation map defined by (p, d) as above. Let

$$h_p = (\Sigma^k p) \circ \eta_p: \Sigma^k M_+ \rightarrow \Sigma^k M_+$$

be the spectral cobordism invariant.

We are now in a position to state and prove the main technical theorem which verifies the condition 2.6 for any double cover over a manifold, and a fortiori helps in proving our main Theorem 1.2 which strengthens Theorem 2.7 of Browder.

Theorem 3.1 (Main technical theorem).

- (a) $h_p^*(H^*(\Sigma^k(M^{2q})_+; \mathbb{Z}/2)) = 0$.
- (b) $k(N)$, the Kervaire invariant of N with respect to η_p , is defined and is zero.

Proof. Consider the Gysin sequence for p (all cohomology groups will be with $\mathbb{Z}/2$ -coefficients):

$$\dots \rightarrow H^{q-1}(M) \xrightarrow{\psi} H^q(M) \xrightarrow{p^*} H^q(N) \xrightarrow{\rho^*} H^q(M) \rightarrow \dots, \tag{1(i)}$$

where $\psi(x) = w \cup x$, w being the characteristic class of the S^0 -bundle p .

We first prove the following analogue of [3, (III.4.1)].

Lemma 3.2. $\eta_p^* \circ \Sigma^* = \Sigma^* \circ \rho^*$.

Proof. We first establish that

$$[M] \cap \rho^*(x) = p_*([N] \cap x). \tag{1}$$

Let $M(p)$ be the mapping cylinder of p . Let $v \in H_{2r+1}(M(p), N) \cong \mathbb{Z}/2$ be the generator. Let $U \in H^1(M(p), N)$ be the Thom class of p , then

$$[M] \cap \rho^*(x) = (v \cap U) \cap \rho^*(x) = v \cap (\rho^*(x) \cup U) = v \cap \delta^*(x),$$

where δ^* is given by the diagram:

$$\begin{array}{ccc} H^*(N) & \xrightarrow{\rho^*} & H^*(M) \\ & \searrow \delta^* & \swarrow \cong \cup U \\ & & H^{*+1}(M(p), N) \end{array} \tag{2(i)}$$

Now $v \cap \delta^*(x) = i_*(\partial v \cap x)$, where $i_* : H_*(N) \rightarrow H_*(M(p))$ and $\partial v = [N]$.

In the diagram:

$$\begin{array}{ccc} N & \xrightarrow{i} & M(p) \\ & \searrow p & \downarrow j \\ & & M \end{array} \tag{2(ii)}$$

j is a homotopy equivalence, therefore, we have

$$i_*(\partial v \cap x) = p_*([N] \cap x).$$

So Eq. (1) is established.

Now, let the double covering $p: N \rightarrow M$ be covered by a bundle map $d = (dp)^{-1} : \tau_N^{-1} \rightarrow \tau_M^{-1}$. The map d induces on Thom spaces the map $T(d) : T(\tau_N^{-1}) \rightarrow T(\tau_M^{-1})$,

and its Spanier–Whitehead $(m + k + s)$ -dual map $\eta_p : \Sigma^s M_+ \rightarrow \Sigma^s N_+$, and we have the homotopy commutative rectangle:

$$\begin{array}{ccc} S^{m+k+s} & \xrightarrow{\gamma} & T(\tau_N^{-1}) \wedge \Sigma^s N_+ \\ \gamma' \downarrow & & \downarrow T(d) \wedge 1 \\ T(\tau_M^{-1}) \wedge \Sigma^s M_+ & \xrightarrow{1 \wedge \eta_p} & T(\tau_M^{-1}) \wedge \Sigma^s N_+, \end{array}$$

where $\gamma_*(i) \cap U_{\tau_N^{-1}} \cap U = \Delta_*[N]$, and $\gamma'_*(i) \cap U_{\tau_M^{-1}} \cap U' = \Delta'_*[M]$, $i \in H_{m+k+s}(S^{m+k+s}; \mathbb{Z}/2)$; $U \in H^s(\Sigma^s N_+; \mathbb{Z}/2)$, $U' \in H^s(\Sigma^s M_+; \mathbb{Z}/2)$ are the Thom classes over the respective trivial bundles, and Δ, Δ' are the respective diagonal maps. It follows from [3, (I.4.15)] that

$$\gamma'_*(i)/\eta_p^*(x \cup U) = T(d)_*(\gamma_*(i)/(x \cup U)). \tag{2}$$

But

$$(\gamma_*(i)/(x \cup U)) \cap U_{\tau_N^{-1}} = (\gamma_*(i) \cap U_{\tau_N^{-1}} \cap U)/x = \Delta_*[N]/x = [N] \cap x,$$

and as $U_{\tau_N^{-1}} = T(d)^*U_{\tau_M^{-1}}$, so

$$(T(d)_*(\gamma_*(i)/(x \cup U))) \cap U_{\tau_M^{-1}} = p_*(\gamma_*(i)/(x \cup U) \cap U_{\tau_N^{-1}}) = p_*([N] \cap x). \tag{3}$$

In a similar manner if $\eta_p^*(x \cup U) = y \cup U'$, it follows that

$$\begin{aligned} (\gamma'_*(i)/\eta_p^*(x \cup U)) \cap U_{\tau_M^{-1}} &= (\gamma'_*(i)/(y \cup U')) \cap U_{\tau_M^{-1}} \\ &= \gamma'_*(i) \cap U_{\tau_M^{-1}} \cap U'/y = \Delta'_*[M]/y = [M] \cap y. \end{aligned} \tag{4}$$

Hence from Eqs. (2), (3), and (4) we get

$$[M] \cap (\Sigma^{*-1} \circ \eta_p^* \circ \Sigma^*(x)) = p_*([N] \cap x). \tag{5}$$

Now combining Eqs. (1) and (5) we get

$$(\Sigma^{*-1} \circ \eta_p^* \circ \Sigma^*(x)) = \rho^*(x).$$

Thus $\eta_p^* \circ \Sigma^* = \Sigma^* \circ \rho^*$, and Lemma 3.2 is established. \square

From Lemma 3.2 it follows that $\ker(\eta_p^* \circ \Sigma^*) = \ker \rho^*$ and also that

$$h_p^* = \eta_p^* \circ (\Sigma^k p)^* = \eta_p^* \circ \Sigma^* \circ p^* \circ \Sigma^{*-1} = \Sigma^* \circ \rho^* \circ p^* \circ \Sigma^{*-1} = 0.$$

So Theorem 3.1(a) is proved.

We need to analyse $k(N)$, which is defined by the quadratic form:

$$\Psi : [\ker(\eta_p^* \circ \Sigma^k)]^r \rightarrow \mathbb{Z}/2; \quad \Psi(x) \stackrel{\text{def}}{=} [Sq_{(\Sigma^k \phi) \circ \eta_p}^{r+1}(\Sigma^k \iota)] [\Sigma^k M_+],$$

where $\phi^*(\iota) = x, \iota$ being the fundamental class of $K(\mathbb{Z}/2, r)$, and Sq_0^{r+1} is the functional Steenrod square. Now

$$[\ker(\eta_p^* \circ \Sigma^k)]^r = [\ker \rho^*]^r = [\text{Im } p^*]^r = [\text{Im } i^*]^r,$$

where the first equality follows from the lemma and the last equality from 2(ii). On the other hand $N \xrightarrow{i} M(p)$ is the Wu($r + 1$)-boundary of $M(p)$ (p being a double cover), therefore $\Psi(\text{Im } i^*) = 0$ (see [1, (1.8)]). Thus $\Psi(x) = 0 \forall x \in [\ker(\eta_p^* \circ \Sigma^k)]^r$. Therefore $k(N)$ is defined and is 0. This proves Theorem 3.1(b), completing the proof of Theorem 3.1. \square

Proof of the Main Theorem 1.2. The Main Theorem 1.2 follows from the Main Technical Theorem 3.1 in exactly the same way as Theorem 2.7 follows from the Assumption 2.6. \square

We have, therefore strengthened the Browder's Theorem 2.7.

4. Splitting invariants along embedded product of real projective spaces

Now, let $f: N \rightarrow \mathbb{R}P^{4q+1}$, q sufficiently large, be a homotopy equivalence. Taking $\xi^k = (f^{-1})^*(v_N)$ we may define a normal map (f, b) . Making f transverse regular on $i(\mathbb{R}P^r \times \mathbb{R}P^s) \subseteq \mathbb{R}P^{4q+1}$ we get a normal map $(f_{r,s}, b_{r,s})$ with

$$f_{r,s}: f^{-1}(i(\mathbb{R}P^r \times \mathbb{R}P^s)) \rightarrow i(\mathbb{R}P^r \times \mathbb{R}P^s),$$

and $b_{r,s}$ induced by b and $f|(\text{tubular neighbourhood of } f^{-1}(i(\mathbb{R}P^r \times \mathbb{R}P^s)) \text{ in } N)$.

Definition 4.1. Let $k_{r,s}(N) \stackrel{\text{def}}{=} k(f^{-1}(i(\mathbb{R}P^r \times \mathbb{R}P^s))) \in \mathbb{Z}/2$.

The following results can be proved in the same way as [2, Theorems 4.12, 4.15 and Corollary 4.16] once we appeal to our main technical Lemma 3.1 and Theorem 1.2.

Theorem 4.2. $k_{r,s}$ depends only on the piecewise linear (topological) equivalence class of N , in particular, it is independent of the choice of the homotopy equivalence f or of the bundle map b .

Theorem 4.3. Suppose a matrix $K = ((k_{r,s}))$, r, s even, $r + s \leq 4q + 1 - rs$ occurs as $k_{r,s} = k_{r,s}(N)$ for a smooth (pl. or top.) homotopy projective space N^{4q+1} . Then the matrix $K' = ((k_{r,s} + 1))$ will also occur for some smooth (pl. or top.) manifold N' .

Now we are in a position to prove Theorem 1.3 (main application 1).

Proof of Theorem 1.3. For each r, s with $r + s < 4q + 1 - rs$, $r, s > 1$, we can define a codimension $rs (> 3)$ surgery problem (see [8, Section 7]) with base map

$$(F, F_{r,s}): (N', N_{r,s}) \rightarrow (\mathbb{R}P^{4q+1}, i(\mathbb{R}P^r \times \mathbb{R}P^s)),$$

where F , as defined in the introduction, has been moved to become transversal to $i(\mathbb{R}P^r \times \mathbb{R}P^s)$ and $N_{r,s} = F^{-1}(i(\mathbb{R}P^r \times \mathbb{R}P^s))$. Since F is already a homotopy equivalence, the

surgery obstruction of making $(F, F_{r,s})$ normally cobordant to $(\bar{F}, \bar{F}_{r,s})$ with both \bar{F} and $\bar{F}_{r,s}$ simultaneously homotopy equivalence is given by

$$\lambda(F, F_{r,s}) = \lambda(F_{r,s}) \in L_{r+s}(\mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}/2]^\pm)$$

(see [11, Section 11], or [8, Proposition 7.2.2]).

We compute $\lambda(F_{r,s})$ for each r, s with $r + s < 4q + 1 - rs$, $r, s > 1$.

Case 1. $(r + s)$ is odd. In this case either r or s is even. Therefore $\mathbb{R}P^r \times \mathbb{R}P^s$ is nonorientable, and by [12], $L_{r+s}(\mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}/2]^-) = 0$. So $\lambda(F_{r,s}) = 0$.

Case 2. $r + s \equiv 0 \pmod{4}$, r, s odd. In this case $\mathbb{R}P^r \times \mathbb{R}P^s$ is orientable. So by [4, Theorem A(a)]

$$\lambda(F_{r,s}) = \text{Index}(h) + \kappa_2(c_*ARF_2(h)).$$

Here and in what follows, h represents the normal invariant corresponding to the relevant normal maps (i.e., a map to G/TOP from the target manifold of the relevant normal map). But from [4, p. 352] one notes that $\forall i \geq 2$, $\text{Im } \kappa_i$ is contained in the image of

$$H^{i+1}(\mathbb{Z}/2; \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}/2])) \rightarrow L_{r+s}(\mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}/2]^+),$$

which is zero since $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}/2]) = 0$ (see [6, p. 76]). So $\lambda(F_{r,s}) = \text{Index}(h)$, the change in index of the surgery problem given by $F_{r,s}$. By the normal cobordism invariance of the index we have

$$\text{Index}(h) = \text{Index}(n(\mathbb{R}P^r \times \mathbb{R}P^s) \cup (-(n-1)/2)M \rightarrow \mathbb{R}P^r \times \mathbb{R}P^s)$$

(this map being normally cobordant to $F_{r,s}$). So

$$\text{Index}(h) = (n-1)\text{Index}(\mathbb{R}P^r \times \mathbb{R}P^s) + (-(n-1)/2)\text{Index}(M),$$

by additivity of the index. Therefore, $\text{Index}(h) = 0 + 0 = 0$. Here we have used the fact that the middle dimensional integral homology of $\mathbb{R}P^r \times \mathbb{R}P^s$, $r + s \equiv 0 \pmod{4}$, r, s odd, and its double cover M are torsion.

Case 3. $r + s \equiv 2 \pmod{4}$, r, s odd. In this case $\mathbb{R}P^r \times \mathbb{R}P^s$ is orientable. So by [4, Theorem A(c)] we have

$$\lambda(F_{r,s}) = ARF(h) + \kappa_4^U \left\{ c_* \left(\sum_{r \geq 0} s_r (ARF_{2r+2}(h)) \right) \right\}.$$

By the same reasoning as in the last case this formula reduces to $\lambda(F_{r,s}) = ARF(h)$. Note first that the Kervaire invariant $k(\mathbb{R}P^r \times \mathbb{R}P^s)$ is defined using the identity normal map of $\mathbb{R}P^r \times \mathbb{R}P^s$ and hence it is defined and is zero, and the same is true for $k(-(\mathbb{R}P^r \times \mathbb{R}P^s))$, by Remark 2.2. Now, by Theorem 1.2(a) we get

$$k(n(\mathbb{R}P^r \times \mathbb{R}P^s) \cup (-(n-1)/2)M) = k(n(\mathbb{R}P^r \times \mathbb{R}P^s)).$$

Orientability of $\mathbb{R}P^r \times \mathbb{R}P^s$ gives that $v_i(\mathbb{R}P^r \times \mathbb{R}P^s) = 0 \forall i$ odd (see [9]), and in particular, $v_{(r+s)/2}(\mathbb{R}P^r \times \mathbb{R}P^s) = 0$. This shows that the square of any element of $H^{(r+s)/2}(\mathbb{R}P^r \times \mathbb{R}P^s; \mathbb{Z}/2)$ is zero. So $((c^*(y))^2 = 0, \forall y \in H^{(r+s)/2}(\mathbb{R}P^r \times \mathbb{R}P^s; \mathbb{Z}/2)$, where $c \stackrel{\text{def}}{=} 1_{\mathbb{R}P^r \times \mathbb{R}P^s}$. Hence by Theorem 2.5 $k(2(\mathbb{R}P^r \times \mathbb{R}P^s))$ is defined, and by Theorem 2.3 $k(2(\mathbb{R}P^r \times \mathbb{R}P^s)) = 2k(\mathbb{R}P^r \times \mathbb{R}P^s) = 0$. As $h_{2c}^* = 2h_c^* = 0$, by repeated

application of the Theorem 2.1 we conclude that $k(2n(\mathbb{R}P^r \times \mathbb{R}P^s))$, $n \geq 2$ are all defined and are zero. Combining this fact with Theorem 2.1 taking one of the factors as $-(\mathbb{R}P^r \times \mathbb{R}P^s)$ we get that $k(n(\mathbb{R}P^r \times \mathbb{R}P^s)) = 0$, $\forall n \geq 1$. Hence

$$k(n(\mathbb{R}P^r \times \mathbb{R}P^s) \cup (-(n-1)/2)M) = k(n(\mathbb{R}P^r \times \mathbb{R}P^s)) = 0.$$

Also since $F_{r,s} : N'_{r,s} \rightarrow \mathbb{R}P^r \times \mathbb{R}P^s$ is normally cobordant to

$$n(\mathbb{R}P^r \times \mathbb{R}P^s) \cup (-(n-1)/2)M \rightarrow \mathbb{R}P^r \times \mathbb{R}P^s,$$

we get that $ARF(h) = k_{r,s}(N') = 0$. Therefore $\lambda(F_{r,s}) = 0$.

Case 4. r, s are both even. The surgery obstruction for this case is given by the Kervaire invariant $k_{r,s}(N')$, because the Wall surgery obstruction group

$$L^h_{\text{even}}(\mathbb{Z}/2 \times \mathbb{Z}/2^-) \cong L^h_{\text{even}}(\mathbb{Z}/2^-) = \mathbb{Z}/2,$$

(see [6, Ch. 1 Sec. 2]) and the surgey obstruction is detected by the Arf invariant (see also [11, 13 B.]). As $(v^2_{(r+s)/2}(\mathbb{R}P^r \times \mathbb{R}P^s))[\mathbb{R}P^r \times \mathbb{R}P^s] \neq 0$ from Theorem 1.2 we get that $k_{r,s}(N') = 1$ for the normal map having base map F . So F cannot be made normally cobordant to any map for which $F_{r,s}$ is a homotopy equivalence (see [3]).

Thus we have calculated the splitting invariants $s_{(r,s)}(F, \bar{c})$ in all possible cases.

The proof of Theorem 1.3 (Main application 1) will thus be completed by successively performing surgeries mod rs to make $F_{r,s}$ homotopy equivalences for every r, s with $r + s < 4q + 1 - rs$; $r, s > 1$ and not both r, s even. \square

5. Splitting invariants along embedded real Milnor manifolds $\mathbb{R}H_{r,s}$

Let (f, b) be a normal map into $\mathbb{R}P^{4q+1}$, q sufficiently large, as in the last section. Let $\hat{i} : \mathbb{R}H_{r,s} \rightarrow \mathbb{R}P^{4q+1}$ be the composite of the following embeddings:

$$\mathbb{R}H_{r,s} \hookrightarrow \mathbb{R}P^r \times \mathbb{R}P^s \hookrightarrow \mathbb{R}P^{4q+1},$$

which were defined in the introduction. Making f transverse regular on $\hat{i}(\mathbb{R}H_{r,s}) \subseteq \mathbb{R}P^{4q+1}$ we get a normal map $(\hat{f}_{r,s}, \hat{b}_{r,s})$ with

$$\hat{f}_{r,s} : f^{-1}(\hat{i}(\mathbb{R}H_{r,s})) \rightarrow \hat{i}(\mathbb{R}H_{r,s}),$$

and $\hat{b}_{r,s}$ induced by b and $f|(\text{tubular neighbourhood of } f^{-1}(\hat{i}(\mathbb{R}H_{r,s})) \text{ in } N)$.

Definition 5.1. Let $\hat{k}_{r,s}(N) \stackrel{\text{def}}{=} k(f^{-1}(\hat{i}(\mathbb{R}H_{r,s}))) \in \mathbb{Z}/2$.

The following results can be proved in the same way as [2, Theorems 4.12, 4.15 and Corollary 4.16] once we appeal to our Main Technical Theorem 3.1 and the Theorem 1.2.

Theorem 5.2. $\hat{k}_{r,s}$ depends only on the piecewise linear (topological) equivalence class of N , in particular, it is independent of the choice of the homotopy equivalence f or of the bundle map b .

Theorem 5.3. *Suppose a matrix $\widehat{K} = ((\widehat{k}_{r,s}))$, $r \geq s > 2$, r odd, s even, $r + s \leq 4q + 1 - rs$ occurs as $\widehat{k}_{r,s} = \widehat{k}_{r,s}(N)$ for a smooth (pl. or top.) homotopy projective space N^{4q+1} . Then the matrix $\widehat{K}' = ((\widehat{k}_{r,s} + 1))$ will also occur for some smooth (pl. or top.) manifold N' .*

Now we are in a position to prove the Theorem 1.4 (main application 2).

Proof of Theorem 1.4. For each r, s with $r + s < 4q + 1 - rs$, $r \geq s > 2$, we can define a codimension $rs + 1 (> 3)$ surgery problem (see [8, Section 7]) with base map

$$(F, \widehat{F}_{r,s}) : (N', \widehat{N}_{r,s}) \rightarrow (\mathbb{R}P^{4q+1}, \widehat{i}(\mathbb{R}H_{r,s})),$$

where F , as defined in the introduction, has been moved to become transversal to $\widehat{i}(\mathbb{R}H_{r,s})$ and $\widehat{N}_{r,s} = F^{-1}(\widehat{i}(\mathbb{R}H_{r,s}))$. Since F is already a homotopy equivalence, the surgery obstruction of making $(F, \widehat{F}_{r,s})$ normally cobordant to $(\overline{F}, \overline{F}_{r,s})$ with both \overline{F} and $\overline{F}_{r,s}$ simultaneously homotopy equivalence is given by

$$\lambda(F, \widehat{F}_{r,s}) = \lambda(\widehat{F}_{r,s}) \in L_{r+s-1}(\mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}/2]^\pm)$$

(see [11, Section 11], or [8, Proposition 7.2.2]).

We compute $\lambda(\widehat{F}_{r,s})$ for each r, s with $r + s < 4q + 1 - rs$, $r \geq s > 2$.

Recall [7, Section 1] that the real Milnor manifold defined in the introduction, $\mathbb{R}H_{r,s}$ with $r \geq s$, can be written as the total space of a fibre bundle

$$\mathbb{R}P^{r-1} \xrightarrow{\text{incl}} \mathbb{R}H_{r,s} \xrightarrow{\text{proj}} \mathbb{R}P^s$$

with fibre $\mathbb{R}P^{r-1}$. Hence using the product formula for the mod 2 Euler characteristics we get that

$$\chi(\mathbb{R}H_{r,s}) = \chi(\mathbb{R}P^{r-1})\chi(\mathbb{R}P^s) = \begin{cases} 1 & \text{if } s \text{ is even, and } r \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover the top Stiefel–Whitney class $w_{r+s-1}(\mathbb{R}H_{r,s}) \neq 0$ if and only if $\chi(\mathbb{R}H_{r,s}) \neq 0$, so we get that

$$w_{r+s-1}(\mathbb{R}H_{r,s}) = \begin{cases} 1 & \text{if } s \text{ is even, and } r \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

So, for $r + s - 1 \equiv 0 \pmod{2}$

$$\begin{aligned} v_{(r+s-1)/2}^2(\mathbb{R}H_{r,s}) &= w_{r+s-1}(\mathbb{R}H_{r,s}) \\ &= \begin{cases} 1 & \text{if } s \text{ is even, and } r \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We recall some more properties of $\mathbb{R}H_{r,s}$.

From the universal coefficient theorem and the mod 2 Leray–Hirsch theorem we get

$$H^1(\mathbb{R}H_{r,s}; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \cong \text{Hom}(H_1(\mathbb{R}H_{r,s}), \mathbb{Z}/2).$$

So, this fact and the homotopy exact sequence of the fibre bundle

$$\mathbb{R}P^{r-1} \xrightarrow{\text{incl}} \mathbb{R}H_{r,s} \xrightarrow{\text{proj}} \mathbb{R}P^s$$

gives that the fundamental group $\pi_1(\mathbb{R}H_{r,s}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, provided $r > 2, s \geq 2$, which are satisfied in our case. Also note that (see [10, pp. 79–80])

$$w_1(\mathbb{R}H_{r,s}) = (r + 1)\pi_1^*a + (s + 1)\pi_2^*b + \sigma_1(l),$$

where $\pi_i : \mathbb{R}P^{r_1} \times \mathbb{R}P^{r_2} \rightarrow \mathbb{R}P^{r_i}$ is the projection to the i th factor, $i = 1, 2$, and

$$\sigma(l) \stackrel{\text{def}}{=} \pi_1^*a + \pi_2^*b.$$

Hence

$$w_1(\mathbb{R}H_{r,s}) = r\pi_1^*a + s\pi_2^*b = r(a \otimes 1) + s(1 \otimes b),$$

where $a \in H^1(\mathbb{R}P^r; \mathbb{Z}/2)$, and $b \in H^1(\mathbb{R}P^s; \mathbb{Z}/2)$ are the generators. Thus

$$w_1(\mathbb{R}H_{r,s}) = \begin{cases} 0 & \text{if } s \text{ and } r \text{ are both even,} \\ \neq 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\mathbb{R}H_{r,s} \text{ is } \begin{cases} \text{orientable} & \text{if } s \text{ and } r \text{ are both even,} \\ \text{nonorientable} & \text{otherwise.} \end{cases}$$

Now we consider the following cases. We will always take $r \geq s > 2$.

Case 1. $\dim(\mathbb{R}H_{r,s}) = r + s - 1 \equiv 1 \pmod{4}$. In this case the Wall surgery obstruction groups are given by

$$L_{r+s-1}(\pi_1(\mathbb{R}H_{r,s}), w) = \begin{cases} L_{r+s-1}(\mathbb{Z}(\mathbb{Z}/2 \times \mathbb{Z}/2)^+) & \text{if } r \text{ and } s \text{ are even,} \\ L_{r+s-1}(\mathbb{Z}(\mathbb{Z}/2 \times \mathbb{Z}/2)^-) & \text{if } r \text{ and } s \text{ are odd} \end{cases} = 0,$$

(see [5, p. 472]). So the surgery obstruction $\lambda(\widehat{F}_{r,s}) = 0$.

Case 2. $\dim(\mathbb{R}H_{r,s}) = r + s - 1 \equiv 0 \pmod{2}$, r is odd, and s is even. In this case $\mathbb{R}H_{r,s}$ is nonorientable, and

$$\begin{aligned} L_{r+s-1}(\pi_1(\mathbb{R}H_{r,s}), w) &= L_{r+s-1}(\mathbb{Z}(\mathbb{Z}/2 \times \mathbb{Z}/2)^-) \\ &\cong L_{r+s-1}(\mathbb{Z}(\mathbb{Z}/2^-)) = \mathbb{Z}/2, \end{aligned}$$

(see [5]), and the surgery obstruction $\lambda(\widehat{F}_{r,s})$ is detected by the Kervaire (Arf) invariant (see [11, 13 B], [6, Chapter 1, Section 2]). Now, as in the given case

$$v_{(r+s-1)/2}^2(\mathbb{R}H_{r,s})[\mathbb{R}H_{r,s}] \neq 0,$$

it follows from Theorem 1.2 that the surgery obstruction

$$\lambda(\widehat{F}_{r,s}) = \hat{k}_{r,s}(N') = 1.$$

Case 3. $\dim(\mathbb{R}H_{r,s}) = r + s - 1 \equiv 0 \pmod{2}$, r is even, and s is odd. Surgery obstruction groups are same as the last case and surgery obstruction is detected by Kervaire (Arf) invariant as in the last case. Now, as in the given case

$$v_{(r+s-1)/2}^2(\mathbb{R}H_{r,s})[\mathbb{R}H_{r,s}] = 0,$$

it follows using exactly similar arguments as in the proof of theorem ((1.3); case 3) of the last section that the surgery obstruction

$$\lambda(\widehat{F}_{r,s}) = \hat{k}_{r,s}(N') = 0.$$

Case 4. $\dim(\mathbb{R}H_{r,s}) = r + s - 1 \equiv 3 \pmod{4}$, $r \geq s > 2$.

In this case the Wall surgery obstruction groups are given by

$$\begin{aligned} & L_{r+s-1}(\pi_1(\mathbb{R}H_{r,s}), w) \\ &= \begin{cases} L_{r+s-1}(\mathbb{Z}/2 \times \mathbb{Z}/2)^+ & \text{if } r \text{ and } s \text{ are even,} \\ L_{r+s-1}(\mathbb{Z}/2 \times \mathbb{Z}/2)^- & \text{if } r \text{ and } s \text{ are odd,} \end{cases} \\ &= \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } r \text{ and } s \text{ are even,} \\ 0 & \text{if } r \text{ and } s \text{ are odd.} \end{cases} \end{aligned}$$

(See [5, p. 472].)

So, $\lambda(\widehat{F}_{r,s}) = 0$ if r and s are odd.

Therefore, we have only to consider the case when r and s are even (and of course $r + s - 1 \equiv 3 \pmod{4}$).

Let us reconsider the fibre bundle

$$\mathbb{R}P^{r-1} \xrightarrow{\text{incl}} \mathbb{R}H_{r,s} \xrightarrow{\text{proj}} \mathbb{R}P^s.$$

Consider the restriction of this fibre bundle to $\mathbb{R}P^{s-1} \subseteq \mathbb{R}P^s$, that is

$$\mathbb{R}P^{r-1} \xrightarrow{\text{incl}} (\text{proj})^{-1}(\mathbb{R}P^{s-1}) \xrightarrow{\text{proj}} \mathbb{R}P^{s-1}.$$

By definition $(\text{proj})^{-1}(\mathbb{R}P^{s-1}) = \mathbb{R}H_{r,s-1} \subseteq \mathbb{R}H_{r,s}$, and $\dim \mathbb{R}H_{r,s-1} = r + s - 2 \equiv 2 \pmod{4}$. Moreover, $\mathbb{R}P^s = \mathbb{R}P^{s-1} \cup_q D^s$ as an attaching space, where $q : S^{s-1} \rightarrow \mathbb{R}P^{s-1}$ is the quotient (identification) map, so the fibre bundle

$$\mathbb{R}P^{r-1} \xrightarrow{\text{incl}} \mathbb{R}H_{r,s} \xrightarrow{\text{proj}} \mathbb{R}P^s$$

is obtained as the fibre-wise identification:

$$\mathbb{R}P^{r-1} \xrightarrow{\text{incl}} \mathbb{R}H_{r,s} = \mathbb{R}H_{r,s-1} \cup_{q \times 1} (D^s \times \mathbb{R}P^{r-1}) \xrightarrow{\text{proj}} \mathbb{R}P^{s-1} \cup_q D^s = \mathbb{R}P^s.$$

Let U be a tubular neighbourhood of $\mathbb{R}H_{r,s-1}$ in $\mathbb{R}H_{r,s}$. Then $(U, \partial U)$ is an $(r + s - 1)$ -dimensional manifold with boundary contained in $\mathbb{R}H_{r,s}$, and $(\mathbb{R}H_{r,s} \setminus U, \partial(\mathbb{R}H_{r,s} \setminus U))$ is homotopy equivalent to $(D^s \times \mathbb{R}P^{r-1}, \partial(D^s \times \mathbb{R}P^{r-1}))$, and

$$\mathbb{R}H_{r,s} = U \cup_{\partial U = \partial(D^s \times \mathbb{R}P^{r-1})} D^s \times \mathbb{R}P^{r-1}.$$

Now $F|_{F^{-1}(\widehat{i}(U, \partial U))} : F^{-1}(\widehat{i}(U, \partial U)) \rightarrow \widehat{i}(U, \partial U)$ is homotopy equivalence if and only if $F|_{\widehat{N}_{r,s-1}} : \widehat{N}_{r,s-1} \rightarrow \widehat{i}(\mathbb{R}H_{r,s-1})$ is homotopy equivalence. But by the last case 3 we have $\lambda(F|_{\widehat{N}_{r,s-1}}) = \lambda(\widehat{F}_{r,s-1}) = 0$, so we can choose F such that $\widehat{F}_{r,s-1} = F|_{\widehat{N}_{r,s-1}}$ is a homotopy equivalence and therefore $F|_{F^{-1}(\widehat{i}(U, \partial U))}$ can be assumed to be a homotopy equivalence.

Note also that by [4, Theorem A]

$$\lambda(F|_{F^{-1}(\widehat{i}(\partial(D^s \times \mathbb{R}P^{r-1})))}) = ARF(h) + \kappa_4 \left\{ c_* \sum_{t>0} s_t(ARF_{2t+2}(h)) \right\}.$$

Now $\text{Im } \kappa_4 = 0$ by the reason given in the proof of case 2 of Theorem 1.3 of the last section, and

$$ARF(h) = (V_{\partial(D^s \times \mathbb{R}P^{r-1})}^2 \cup h^*(k)) \cap [\partial(D^s \times \mathbb{R}P^{r-1})] = 0,$$

because

$$V_{\partial(D^s \times \mathbb{R}P^{r-1})}^2 \cup h^*(k) = 0 \in H^{r+s-2}(S^{s-1} \times \mathbb{R}P^{r-1}; \mathbb{Z}/2),$$

as $r \geq s > 2$, and the contribution of both the characteristic classes $V_{\partial(D^s \times \mathbb{R}P^{r-1})}^2$ and $h^*(k)$ are zero in $H^{s-1}(S^{s-1}; \mathbb{Z}/2)$. Therefore,

$$\lambda(F|_{F^{-1}(\hat{i}(\partial(D^s \times \mathbb{R}P^{r-1})))}) = 0.$$

So we can assume that $F|_{F^{-1}(\hat{i}(\partial(D^s \times \mathbb{R}P^{r-1})))}$ is ambiently normally cobordant to a homotopy equivalence.

Next appealing again to [4, Theorem A] we get that

$$\lambda(F|_{F^{-1}(\hat{i}(D^s \times \mathbb{R}P^{r-1}, \partial(D^s \times \mathbb{R}P^{r-1})))}) = \kappa_1(c_* ARF_1(h)),$$

where

$$\begin{aligned} ARF_1(h) &= (V_{D^s \times \mathbb{R}P^{r-1}}^2 \cup h^*(k)) \\ &\cap [D^s \times \mathbb{R}P^{r-1}, \partial(D^s \times \mathbb{R}P^{r-1})] \in H_1(D^s \times \mathbb{R}P^{r-1}; \mathbb{Z}/2). \end{aligned}$$

But

$$V_{D^s \times \mathbb{R}P^{r-1}}^2 \cup h^*(k) \in H^{r+s-2}(D^s \times \mathbb{R}P^{r-1}; \mathbb{Z}/2) = 0,$$

because $r \geq s > 2$. Therefore, $ARF_1(h) = 0$, and hence

$$\lambda(F|_{F^{-1}(\hat{i}(D^s \times \mathbb{R}P^{r-1}, \partial(D^s \times \mathbb{R}P^{r-1})))}) = 0.$$

Now we can appeal to the addition theorem of surgery obstructions (see [3, (II.1.4)]) (we have verified all the hypotheses of the theorem) to get

$$\begin{aligned} \lambda(\widehat{F}_{r,s}) &= \lambda(\widehat{F}_{r,s}|_{F^{-1}(\hat{i}(U, \partial U))}) + \lambda(\widehat{F}_{r,s}|_{F^{-1}(\hat{i}(D^s \times \mathbb{R}P^{r-1}, \partial(D^s \times \mathbb{R}P^{r-1})))}) \\ &= 0 + 0 = 0, \end{aligned}$$

by our above calculations.

Thus we have obtained the splitting invariants $\hat{s}_{(r,s)}(F, \bar{c})$ in all possible cases.

The proof of Theorem 1.4 (main application 2) will thus be completed by successively performing surgeries mod $(rs + 1)$ to make $\widehat{F}_{r,s}$ homotopy equivalences for every r, s with $r + s < 4q + 1 - rs$; $r \geq s > 2$ except for those r, s which satisfy r odd, s even. \square

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