DARK ENERGY AND THE ACCELERATING UNIVERSE

BY

BHARAT BORAH

DEPARTMENT OF MATHEMATICS

NORTH-EASTERN HILL UNIVERSITY

SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF

MASTER OF PHILOSOPHY

To

NORTH-EASTERN HILL UNIVERSITY

SHILLONG - 793022

OCTOBER, 2010

CERTIFICATE

I certify that the dissertation entitled "DARK ENERGY AND THE ACCELERATING UNIVERSE" submitted by Mr. Bharat Borah in partial fulfilment of the requirement of the degree of Master of Philosophy in Mathematics is the outcome of a study undertaken by the candidate.

I certify that the sources from which ideas have been borrowed have been duly referred to.

The material in this dissertation has not been presented for the award of a degree in any university before.

This dissertation may be placed before the examiners for evaluation and necessary formalities. I certify that this dissertation is worthy of consideration by the examiners.

M. Ansari
Supervisor
Department of Mathematics
North-Eastern Hill University
Shillong – 793022
Email: mansari@nehu.ac.in

Place: Shillong.
4^{th} October, 2010.

NORTH-EASTERN HILL UNIVERSITY

October, 2010

DECLARATION

I, Bharat Borah, hereby declare that the subject matter in this dissertation is the record of work done by me, that the contents of this dissertation did not form basis of the award of any previous degree to me or to the best of my knowledge to anybody else, and that the dissertation has not been submitted by me for any research degree in any other university/institute.

This dissertation is being submitted to the North-Eastern Hill University for the degree of Master of Philosophy in Mathematics.

Signature of the Candidate

Countersigned by:

Signature of the Head

Signature of the Supervisor

ACKNOWLEDGEMENT

This work was carried out under the supervison of Dr. M. Ansari, Department of Mathematics, North-Eastern Hill University. I sincerely convey my heartfelt gratitude to him for his excellent guidance and constant inspiration for the completion of my work.

I would like to thank Dr. K. K. Singh, Department of Mathematics, N.E.H.U. for giving me a course as per the requirement of the M. Phil programme and for providing me with lots of help and suggestions.

I also express my gratitude to (L) Prof. S. K. Srivastava for his initial guidance, Prof. H. K. Mukerjee, Prof. M. B. Rege, Dr. P. K. Saikia, Dr. A. K. Das, Dr. S. Dutta, Dr. A. M. Buhphang, Sir A. T. Singh and all other faculty members of the Department of Mathematics, N.E.H.U., for their help and encouragement.

I am very much indebted to all the Research Scholars and the office staffs of Mathematics Department, N.E.H.U., for extending all possible help to me.

I am grateful to all my relatives and friends for their support and for being my constant source of inspiration.

Finally, I am grateful to all my family members, especially my parents for giving me constant encouragement and providing me with unbelievable opportunities to continue my studies.

Also primordial to mention here is God's care, blessings and guidance in all my undertakings.

Bharat Borah

PREFACE

In physical cosmology, astronomy and celestial mechanics, dark energy is a hypothetical form of energy that permeates all of space and tends to increase the rate of expansion of the universe. Dark energy is the most popular way to explain recent observations that the universe appears to be expanding at an accelarating rate [1, 2]. In general relativity, the evolution of the expansion rate is parameterized by the cosmological equation of state [3]. Measuring the equation of state of dark energy is one of the biggest efforts in observational cosmology today. Dark energy has been used as a crucial ingredient in a recent attempt to formulate a cyclic model [4] for the universe.

The discovery in 1998 that the universe is not only expanding it is accelerating have caused a sea change in the thinking about the universe. The 16 type Ia Supernovae [5], observed by Hubble Space Telescope(HST), further modifies earlier astronomical results and provides conclusive evidence supporting deceleration prior to cosmic acceleration caused by dark energy in the recent past, which is realized with negative pressure and positive energy density violating strong energy condition (SEC). Violation of SEC gives reverse gravitational effect. It is due to this effect, universe gets a jerk and transition from deceleration to acceleration takes place.

The dicovery of cosmic acceleration is arguably one of the most important developments in modern cosmology. But the physical origin of cosmic acceleration remains a deep mystery. According to General Relativity (GR), if the universe is filled with ordinary matter or radiation, the two known constituents of the universe, gravity should lead to a slowing of the expansion. Since the expansion is speeding up, we are faced with two possibilities, either of which would have profound implications for our understanding of the cosmos and of the laws of physics. The first is that 75% of the universe exists in a new form with large negative pressure, called dark energy. The other possibility is that General Relativity breaks down on cosmological scales and must be replaced with a more complete theory of gravity.

Through a tangled history, now the question arises what is the source of dark energy. So it was that cosmological constant Λ could provide a possible source. But this proposal faced a problem that, in the beginning of the universe, it has a very high value. For example at Plank scale it is $\sim 10^{75} GeV^4$ and at QCD scale it has $\sim 10^{-6} GeV^4$. But in the present universe, dark energy density is found to be around $2 \times 10^{-47} GeV^4$. The question then remains, why has Λ got the value it has today ? To satisfy these extreme values of dark energy, it was proposed that a dynamical model of Λ should be introduced.

Rather than dealing directly with the cosmological constant a number of alternative routes have been proposed which skirt around this thorny issue [6, 7, 8]. Cosmologists proposed scalar fields are also the proposed form of dark energy [9], whose energy density can vary in time and space. Contributions from scalar fields, that are constant in space are usually also included in cosmological constant. An incomplete list of scalar field model includes: Quintessence models [10], K-essence model [11, 12], tachyon model [13], Phantom model [14] and dilatonic models. Quintessence is a scalar for which SEC is violated with equation of parameter w having the range $-\frac{1}{3} > w > -1$ (with w being the ratio of pressure and energy density). Later on, it was found from string theory [15] that, at the low energy level, tachyon follows the conditions of dark energy. In 2002, Caldwell argued that observations support the condition $w < -1$ in a better way. This observation led to the proposal of phantom dark energy. It was found that phantom dark energy landed with big-rip problem [10] , which is a finite time future singularity. So there were many attempts to avoid this problem.

There exists another interesting class of dark energy involving a fluid known as a Chaplygin gas [16] . This model is termed as Chaplygin gas model. This fluid also leads to the acceleration of the universe at late times. Remarkably, the Chaplygin gas appears like pressure-less dust at early times and like a cosmological constant during very late times, thus leading to an accelerated expansion. Moreover, curvature induced dark energy causing present acceleration is also a very interesting subject in this arena.

The open question concerning the nature of dark energy and dark matter is certainly one of the most compelling issues of modern astrophysics, that demands new and better observations in order to be answered. Nojiri and Odintsov [17] has discussed curvature induced dark energy. They have taken nonlinear terms of curvature as lagrangian for dark energy. But it is more appropriate if theory is developed where dark energy radiation and matter terms emerge spontaneously from the undertaken theory. Phantom dark energy has thermodynamical problem that either temperature becomes negative or entropy gets negative. It is important to see whether Phantom dark energy emerging from higher derivative gravity is free from this problem.

This thesis contains a systematic review of all developments about dark energy mentioned above. We try to review in detail a number of approaches that have been adopted to explain the remarkable observation of our accelerating universe in terms of dark energy. The dissertation has been divided into five chapters.

The first chapter is introductory. Here we have discussed some basic concepts from cosmology, which contains cosmological principle, Hubble's law, Friedmann Robertson Walker model and some important relations of cosmological parameters.

In chapter 2, we have studied the hot Big-Bang scenario of the universe, observational evidence for dark energy and briefly about the developments of dark energy model.

Chapter 3 is devoted to dark energy and cosmological constant. Here we try to give a details about the cosmological constant and the models evolving cosmological constant.

In chapter 4, we have studied some scalar field models of dark energy, such as Quintessence, K-essence, Tachyon scalar, Phantom scalar and Chaplygin gas model.

Finally, in chapter 5, we have studied in detail the nature of possible future singularities and the future of our universe in presence of cosmological constant Λ.

Contents

Chapter 1

Introduction

Cosmology has made a lot of progress during the last years and has at present a substantial observational and experimental basis that confirm that many aspects of the standard cosmological picture are a good approximation to reality. Still, the empirical basis has not reached the level of precision and accuracy of the Standard Model of particle physics and we cannot talk about a well - established theory of cosmology, in which one can measure the parameters with high precision. With all this, one can say that cosmology is living nowadays a golden epoch and the observational data, which become more and more precise, keeps cosmologists optimistic about establishing a true Standard Cosmological Theory in the near future.

The purpose of cosmology is to study the behaviour of the universe from a global point of view and to describe the physical processes throughout its evolution. The models describing the universe and its evolution are based upon an extrapolation of physical theories whose validity has been locally tested out. The extrapolation is based on the theory of relativity which states that physical laws are co-variant at all times and at every space point.

At large scale the only interaction is gravitation. Earlier attempts (non-relativistic) to understand science of universe through Newtonian mechanics, could not progress due to some fault in Newtonian gravity as (i) it is a static theory and (ii) gravitational field propagation at cosmological scale is doubtful. So the only powerful machinery for this purpose is general relativity and its various modifications from time to time.

After advent of General Relativity, Einstein tried to understand cosmology using his theory of gravitation. But, he had the opinion that universe should be static. So, introducing cosmological constant, he got static solution. In this arena, major breakthrough came when Hubble observed expansion in the universe. So, Einstein's idea of static universe was abandoned and there were various attempts to obtain dynamical models of the universe solving Einstein field equations. Among these models, de-sitter and Friedmann-Robertson-Walker(FRW) models are prominent.

One of the most important issues of modern cosmology concerns the accelerating expansion of the universe, which has been discovered and verified by the type Ia Supernovae (Riess et al. 1998; Perlmutter et al. 1999; Hicken et al. 2009), Cosmic microwave background (CMB)(Spergel et al. 2003) and baryon acoustic oscillation (BAO)(Eisenstein et al. 2005).

The existing mechanisms for cosmic acceleration can be roughly assorted into three kinds (Copeland et al. 2006; Caldwell and kamionkowski 2009):

(a) An exotic energy component with negative pressure, dubbed as dark energy, is introduced in the right hand side of Einstein equation. The nature of the dark energy is still unknown. Dark models include ΛCDM model (Carroll et al. 1992; Riess et al. 1998; Peebles and Ratra 2003), holographic dark energy model (Li 2004; Wu et al. 2008), Chaplygin gas model (Kamenshchick et al. 2001; Benaoum 2002; Zhu 2004; Zhang and Zhu 2006) and some scalar field models, such as Quintessence model (Caldwell et al. 1998), tachyon model (T. Padmanabhan and T.R Choudhury 2002), Phantom model (R.R.Caldwell 2002).

(b) Theory of gravity is modified at the Hubble scale, and the cosmic acceleration is due to gravity, without the help of any exotic negative pressure component. Examples of modified gravity theory include braneworld model (Arkani-Hamed et al. 1998; Dvali et al. 2000; Zhu and Alcaniz 2005), $f(R)$ gravity (Caroll et al. 2004; Nojiri and Odintsov 2003; Song et al. 2007; Atazadeh et al. 2008; Wu and Zhu 2008) and Cardassian model (Freese and Lewis 2002; Sen and Sen 2003; Mosquera Cuesta et al. 2008).

(c) The local inhomogeneity of our universe is used to explain the acceleration (George 2008).

1.1 Cosmological Principle and Hubble's Law

Cosmological principle :

In physical cosmology, the cosmological principle is an assumption, or working hypothesis, about the large scale structure of the cosmos. During the time of Copernicus, much information were not available for the universe except Earth, few stars and planets. So he assumed that the universe would be same from all other planets also as it looked from the Earth. It implies isotropy of the universe at all points. Again, a space which is isotropic at all points, is also homogeneous. Copernicus principle and this result about homogeneity makes the Cosmological principle (CP) (the name given by E.A.Milne), which states that, at a particular time, universe is homogeneous and isotropic. General covariance ensures validity of Cosmological Principle at other times also.

Hubble's law :

In 1929, Edwin Hubble made a very important observation. In his experiment, he estimated the distance to 18 galaxies from the apparent luminosity of their brightest stars and compared these distances with velocities of respective galaxies determined by spectral shift due to Doppler's effect. This observation proved to be a founding stone for cosmology.

According to Doppler's effect, when a source goes away from the observer, frequency of the signal decreases and wavelength increases with distance. Hubble used it for galaxies. Thus if λ is the wavelength of the light signal from the galaxy and $\lambda + d\lambda$ is the wavelength of the same light signal received by the observer then Hubble observed the red-shift as

$$z = \frac{d\lambda}{\lambda}\,. \tag{1.1.1}$$

Suppose that waves of light leave the source at regular intervals dt. If the source is moving with velocity V, it moves a distance Vdt during the time interval dt. So observer receives light signals at regular intervals

$$dt' = dt + \frac{V dt}{c} \,. \tag{1.1.2}$$

Moreover, wavelength is proportional to time. So,

$$\frac{\lambda + d\lambda}{\lambda} = \frac{dt'}{dt} = 1 + \frac{V}{c} \,, \tag{1.1.3}$$

yielding

$$V = cz \,. \tag{1.1.4}$$

The light signals travel a distance D in time interval dt and a distance $D+dD$ during the time interval dt'. So,

$$\frac{dt'}{dt} = \frac{D + dD}{D} = \frac{\lambda + d\lambda}{\lambda} \,, \tag{1.1.5}$$

yielding

$$z = \frac{dD}{D} \,. \tag{1.1.6}$$

this result shows that if $z > 0$, $dD > 0$. Now,

$$V = c\frac{dD}{D} = HdD \,, \tag{1.1.7}$$

with $H = \dfrac{c}{D}$ being the Hubble's constant. It shows that objects, in the universe, are not static, but moving relative to each other. In 1931, this expression was improved. As a result, Hubble was able to verify the proportionality between velocity and distance of galaxies with velocities up to $20,000$ km/s. Later on, it was found that actually H was not a constant, but depends on time giving H^{-1} as age of the universe.

1.2 Framework of Friedmann-Robertson-Walker (FRW) cosmology

For development of a physical theory, we need a model consistent with known facts of the system. All the well developed models of standard cosmology start with two basic assumptions :

(a) The distribution of matter in the universe is homogeneous and isotropic at sufficiently large scales.

(b) The large scale structure of the universe is essentially determined by gravitational interactions and hence can be described by Einstein's theory of gravity.

The geometry of the universe can then be determined via Einstein's equations with the stress tensor of matter $T_k^i(t, X)$ acting as the source. The first assumption determines the kinematics of the universe while the second one determines the dynamics. The most successful model for evolving universe was obtained by Friedmann, in 1922 as well as by Robertson and Walker in 1930, which is called Friedmann-Robertson-Walker (FRW) model. This model was derived by solving the Einstein field euations with the cosmic matter as a non-interacting tiny gas particle behaving like a perfect fluid. This is a non static centrally symmetric homogeneous model obeying the cosmological principle. This model is so successful that the standard hot big-bang theory is based on it. We shall discuss the consequences of the two above assumptions with FRW model in the next two subsections.

1.2.1 Kinematics of the Friedmann-Robertson-Walker (FRW) model

Geometry of a homogeneous and isotropic universe is given by FRW model [18]

$$ds^2 = dt^2 - a^2(t)dX^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \, , \quad (1.2.1)$$

where $a(t)$ is the scale factor or expansion factor with cosmic time t. The co-ordinates r, θ and ϕ are known as co-moving co-ordinates. A freely moving particle comes to rest in these co-ordinates. The constant k is the curvature parameter, describes the geometry of the spatial section of space-time having values

$$k = \begin{cases} +1 & \text{for closed model} \\ 0 & \text{for flat model} \\ -1 & \text{for open model} \end{cases}$$

Defining a new co-ordinate χ through $\chi = (r, \sin^{-1} r, \sinh^{-1} r)$ for $k = (0, +1, -1)$ this line element becomes

$$ds^2 \equiv dt^2 - a^2(t)dX^2 \equiv dt^2 - a^2(t)\left[d\chi^2 + s_k^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right] \, , \quad (1.2.2)$$

where

$$s_k(\chi) = \begin{cases} \sin\chi \, , & k = +1 \\ \chi \, , & k = 0 \\ \sinh\chi \, , & k = -1 \end{cases}$$

In any range of time during which $a(t)$ is a monotonic function of t, one can use a itself as a time co-ordinate. It is also convenient to define a quantity

z, called the redshift, through the relation $a(t) = a_0[1+z(t)]^{-1}$ where a_0 is the present value of the expansion factor. The line element in terms of $[a, \chi, \theta, \phi]$ or $[z, \chi, \theta, \phi]$ is

$$ds^2 = H^{-2}(a)\left(\frac{da}{a}\right)^2 - a^2 dX^2 = \frac{1}{(1+z)^2}\left[H^{-2}(z)dz^2 - dx^2\right] , \qquad (1.2.3)$$

where $H(a) = (\dfrac{\dot{a}}{a})$ is called the Hubble parameter, measures the rate of expansion of the universe.

This equation allows us to draw an important conclusion : The only non trivial metric function in a Friedmann universe is the function $H(a)$ (and the numerical value of k which is encoded in the spatial part of the line element). Hence, any kind of observation based on geometry of the space-time, however complex it may be, will not allow us to determine anything other than this single function $H(a)$. As we shall see, this alone is inadequate to describe the material content of the universe and any attempt to do so will require additional inputs.

Since the geometrical observations often rely on photons received from distant sources, let us consider a photon traveling a distance $r_{em}(z)$ from the time of emission (corresponding to the redshift z) till today. Using the fact that $ds = 0$ for a light ray and the second equality in equation (1.2.3) we find that the distance traveled by light rays is related to the redshift by $dx = H^{-1}(z)dz$. Integrating this relation, we get

$$r_{em}(z) = s_k(\alpha) \quad ; \quad \alpha \equiv \frac{1}{a_0}\int_0^z H^{-1}(z)dz \, . \qquad (1.2.4)$$

All other geometrical distances can be expressed in terms of $r_{em}(z)$. For example, the flux of radiation F received from a source of luminosity L can be expressed in the form $F = \dfrac{L}{(4\pi d_L^2)}$ where

$$d_L(z) = a_0 r_{em}(z)(1+z) = a_0(1+z)s_k(\alpha)\,, \tag{1.2.5}$$

is called the luminosity distance. Similarly, if D is the physical size of an object which subtends an angle δ to the observer, then for small δ we can define an anglular diameter distance d_A through the relation $\delta = \dfrac{D}{d_A}$. The angular diameter distance is given by

$$d_A(z) = a_0 r_{em}(z)(1+z)^{-1}\,, \tag{1.2.6}$$

with $d_L = (1+z)^2 d_A$.

If we can identify some objects (or physical phenomena) at a redshift of z having a characteristic transverse size d, then measuring the angle δ subtended by this object we can determine $d_A(z)$. Similarly, if we have a series of luminous sources at different redshifts having known luminosity L, then by observing the flux from these sources L , one can determine the luminosity distance $d_L(z)$. Determining any of these functions will allow us to use the relations given by equations (1.2.5) [or (1.2.6)] and (1.2.4) to obtain $H^{-1}(z)$. For example, $H^{-1}(z)$ is related to $d_L(z)$ through

$$H^{-1}(z) = \left[1 - \frac{K d_L^2(z)}{a_0^2(1+z)^2}\right]^{-\frac{1}{2}} \frac{d}{dz}\left[\frac{d_L(z)}{1+z}\right] \longrightarrow \frac{d}{dz}\left[\frac{d_L(z)}{1+z}\right]\,, \tag{1.2.7}$$

where second equality holds if the spatial sections of the universe are flat, corresponding to $k=0$; then $d_L(z)$, $d_A(z)$, $r_{em}(z)$ and $H^{-1}(z)$ all contain the (same) maximal amount of information about the geometry.

The function $r_{em}(z)$ also determines the proper volume of the universe between the redshifts z and $z+dz$ subtending a solid angle $d\Omega$ in the sky.

The comoving volume element can be expressed in the form

$$\frac{dV}{dzd\Omega} \propto r_{em}^2 \frac{dr}{dz} \propto \frac{d_L^3}{(1+z)^4} \left[\frac{(1+z)d_L'}{d_L} - 1 \right] , \tag{1.2.8}$$

where the prime denotes derivative with respect to z. Based on this, there has been a suggestion that future observations of the number of dark matter halos as a function of redshift and circular velocities can be used to determine the co-moving volume element to within a few percent accuracy. If this becomes possible, then it will provide an additional handle on constraining the cosmological parameters.

The above discussion illustrates how cosmological observations can be used to determine the metric of the space-time, encoded by the single function $H^{-1}(z)$. This issue is trivial in principle, through enormously complicated in practice because of observational uncertainties in the determination of $d_L(z), d_A(z)$ etc.

1.2.2 Dynamics of the Friedmann-Robertson-Walker (FRW) model

Let us now turn to the second assumption which determines the dynamics of the universe. When several non interacting sources (like radiation, matter, cosmological constant etc) are present in the universe, the total energy momentum tensor which appear on the right hand side of the Einstein's equation will be the sum of the energy momentum tensor for each of these sources.

The differential equations for the scale factor and the matter density follow from Einstein's equations [18].

$$G^\mu_\nu \equiv R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = 8\pi G T^\mu_\nu\,, \tag{1.2.9}$$

where G^μ_ν is the Einstein tensor, and R^μ_ν is the Ricci tensor which depends on the metric and its derivatives, R is the Ricci scalar and T^μ_ν is the energy momentum tensor. In the FRW background given by equation (1.2.1) the curvature terms are given by [19]

$$R^0_0 = \frac{3\ddot{a}}{a}\,, \tag{1.2.10}$$

$$R^i_j = (\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{2k}{a^2})\delta^i_j\,, \tag{1.2.11}$$

$$R = 6(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}). \tag{1.2.12}$$

where a dot denotes a derivative with respect t.

Let us consider an ideal perfect fluid as the source of the energy momentum tensor. Spatial homogeneity and isotropy imply that each T^μ_ν is diagonal and has the form

$$T^\mu_\nu = \mathrm{Diag}(\rho, -p, -p, -p), \tag{1.2.13}$$

where ρ and p are the energy density and the pressure density of the fluid respectively. Then equation (1.2.9) gives two independent equations

$$H^2 \equiv (\frac{\dot{a}}{a})^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2}, \tag{1.2.14}$$

and Raychoudhurie's equation,

$$\dot{H} = -4\pi G(p+\rho) + \frac{k}{a^2}. \tag{1.2.15}$$

where H is the Hubble parameter, ρ and p denotes the total energy density and pressure of all species present in the universe at a given epoch.

The energy momentum tensor with components $T^{\mu}_{\nu}(\mu, \nu = 0, 1, 2, 3)$ is conserved by virtue of the identities $T^{\mu}_{\nu;\mu} = 0$, where semi-colon (;) denotes covariant derivatives leading to the continuity equation

$$\dot{\rho} + 3H(p + \rho) = 0. \tag{1.2.16}$$

Eleminating the $\frac{k}{a^2}$ term from equation (1.2.14) and equation (1.2.15), we obtain

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{1.2.17}$$

Hence the accelerated expansion occurs for $\rho + 3p < 0$. One can rewrite equation (1.2.14) in the form

$$\Omega(t) - 1 = \frac{k}{(aH)^2}\,. \tag{1.2.18}$$

where $\Omega(t) \equiv \frac{\rho(t)}{\rho_c(t)}$ is the dimensionless density parameter and $\rho_c(t) = \frac{3H^2(t)}{8\pi G}$ is the critical density. The matter distribution clearly determines the spatial geometry of our universe, i.e.

$$\Omega > 1 \text{ or } \rho > \rho_c \longrightarrow k = +1, \tag{1.2.19}$$

$$\Omega = 1 \text{ or } \rho = \rho_c \longrightarrow k = 0, \tag{1.2.20}$$

$$\Omega < 1 \text{ or } \rho < \rho_c \longrightarrow k = -1. \tag{1.2.21}$$

Observations have shown that the current universe is very close to a spatially flat geometry ($\Omega \backsimeq 1$) [3]. This is actually a natural result from inflation in the early universe.

1.3 Problems of the standard cosmological model

The standard model of hot big-bang cosmology depends on the assumption of initial conditions. Here by “standard model”, we mean adiabatically expanding radiation-dominated universe described by a FRW matric and the initial conditions are taken as follows. The standard model has a singularity at time $t = 0$. As $t \longrightarrow 0$, the temperature $T \longrightarrow \infty$. So no original problem at the time $t = 0$ can be expressed.

We have planck mass $M_p = \dfrac{1}{\sqrt{G}} = 1.22 \times 10^{19}$GeV, when T is at the order of planck mass or even greater, there will be no meaning adhered to standard model. Near $T \approx 10^{19}$GeV, gravity is expected to be unified with other interactions of nature, for example strong interaction, weak interaction and electromagnetism. Hence to study the effects of gravity separately it is wise to start with a temperature below the planck scale. Due to this reason Guth started his model with temperature $T_0 \approx 10^{17}$GeV. So it will be convenient to set initial conditions of the universe to derive the subsequent evolution. On the contrary, the equation of state for matter at these temperatures can not be practically be deduced, however, various hypothesies can be assumed and conclusion may be derived.

(a) Horizon problem :

From FRW line element given by equation (1.2.3) and using $ds^2 = 0$ for photons, the particle horizon of a light is obtained as

$$d_{PH} = \int_0^{r_H} \frac{dr}{\sqrt{1 - kr^2}} = \int_0^t \frac{dt'}{a(t')} \, , \tag{1.3.1}$$

where the light emitted from a point (r_H, θ_1, ϕ_1) and received at $(0, \theta_1, \phi_1)$.

In the expanding universe, expansion of the universe is realized through the growth of scale factor $a(t)$, so d_{PH} is not the actual distance travelled by the light. The actual distance is given by horizon distance,

$$d_H = a(t) d_{PH} = a(t) \int_0^t \frac{dt'}{a(t')}\,, \tag{1.3.2}$$

and it is called event horizon. Communication through light signal is possible upto this distance. Objects located beyond this distance are causally disconnected.

In the radiation dominated model,

$$d_H = 2t\,, \tag{1.3.3}$$

and

$$\begin{aligned} a(t) &= [\frac{8\pi^3}{45} G]^{1/4} T_0 a_0 t^{1/2} \\ &= [\frac{8\pi^3}{45} G]^{1/4} [\frac{45}{4\pi^2} S_0]^{1/3} t^{1/2} \\ &= 5.06 \times 10^{19} t^{1/2}\,, \end{aligned} \tag{1.3.4}$$

where $S_0 = 1.45 \times 10^{87}$. We have causally disconnected region so long as

$$d_H(t) < a(t). \tag{1.3.5}$$

So , the number of causally disconnected regions can be obtained as ,

$$N_{CD} \;=\; \left[\frac{a(t)}{d_H(t)}\right]^3, \tag{1.3.6}$$

Connecting equations (1.3.3),(1.3.4) and (1.3.6), N_{CD} is obtained to be

$$\begin{aligned} N_{CD} &= \left[2.53 \times 10^{19} t^{-1/2}\right]^3 \\ &\simeq 5 \times 10^{86} \,, \end{aligned} \tag{1.3.7}$$

at Planck time .

After decoupling of matter from radiation, scale factor is given as

$$a(t) = \frac{2}{3}\left[\frac{8\pi G}{3}\rho_0^{(m)} a_0^3\right]^{1/2} t^{2/3} \,. \tag{1.3.8}$$

In this case $d_H(t) = 3t$.

In this phase, the number of causally disconnected regions is

$$N_{CD} = \frac{8}{729}\left[\frac{8\pi G}{3}\rho_0^{(m)} a_0^3\right]^{3/2} t^{-1} \,, \tag{1.3.9}$$

which shows that number of causally disconnected regions is inversely proportional to time .

From the above analysis, we come to know that initially the universe contained many causally disconnected regions but gradually number of disconnected regions decreased . This is the horizon problem . Obviously inhomogeneity is caused by disconnected region , so horizon problem is obtained as homogeneity problem .

(b) Flatness problem : According to the standard cosmology, early universe expands in the radiation-dominated phase with the scale factor

$$a = \left[\frac{8\pi G\rho_0^{(r)} a_0^4}{3}\right]^{1/4} t^{1/2} \,. \tag{1.3.10}$$

Equation (1.3.10) yields the Hubble's parameter (H) as

$$H = \frac{\dot{a}}{a} = \frac{1}{2t} \,. \tag{1.3.11}$$

The standard cosmology expands adiabatically throughout evolution of the universe. So entropy remains constant and heat loss is negligible. As a consequence,

$$aT = a_0 T_0 \,, \tag{1.3.12}$$

where a_0 is the present scale factor and T_0 is the present temperature. Equations (1.3.10) and (1.3.12) lead to

$$T = \left[\frac{45}{8\pi^3 G}\right]^{1/4} t^{-1/2} \,, \tag{1.3.13}$$

Moreover , Friedmann equation (1.2.14) yields

$$\begin{aligned} \frac{\rho}{\rho^{(c)}} &= 1 \pm \frac{|k|}{H^2 a^2} \\ &= 1 \pm \frac{45|k|}{2\pi^3 G a_0^2 T_0^2 T^2} \,, \end{aligned} \tag{1.3.14}$$

where $\rho^{(c)} = \dfrac{3H^2}{8\pi G}$ is the critical density and k is the curvature parameter. The present size of the observable universe is expected to be $a_0 \sim 10^{28}$cm $= 5.08 \times 10^{41} GeV^{-1}$ and $T_0 = 2.7K = 2.33 \times 10^{-13} GeV$.

So at Planck scale i.e at $T = 10^{19} GeV$, equation (1.3.14) yields

$$\frac{\rho}{\rho^{(c)}} = 1 \pm 5.2 \times 10^{-59} |k| \,. \tag{1.3.15}$$

This result shows that we get almost same value whether $k = 0$ or $|k| = 1$. So, it indicates spatial flatness in the begining of the universe. This is the flatness problem.

1.4 Some important relations of cosmological parameters

Cosmology, being a physical theory, needs observational support. Some of the important parameters being observed for cosmological test are given as below.

(a) Cosmological red-shift and scale factor:

Red-shift is one of the very useful parameter of observational cosmology. We often use a red-shift to describe the evolution of the universe. This is related to the fact that light emitted by a stellar object becomes red-shifted due to the expansion of the universe. The wavelength λ increases proportionally to the scale factor a, whose effect can be quantified by the red-shift z, as

$$1 + z = \frac{\lambda_0}{\lambda} = \frac{a_0}{a} \,, \tag{1.4.1}$$

where the subscript zero denotes the quantities given at the present epoch.

(b) Luminosity and modulus of distance relationship : The direct evidence for the current acceleration of the universe is related to the observation of luminosity distances of high redshift supernovae. The apparent magnitude m of the source with an absolute magnitude M is related to the luminosity distance d_L via the relation

$$m - M = 5log_{10}(\frac{d_L}{Mpc}) + 25 \,, \tag{1.4.2}$$

The numerical factors arise because of conventional definitions of m and M in astronomy.

(c) Red-shift and modulus of distance relationship :

An useful relation between red-shift and modulus of distance is obtained as

$$m - M = 25 + 5logz - 5logH_0 + 5log(q_0^2)^{-1} \left[q_0 + (q_0 - 1) \frac{\{(1 + 2q_0 z)^{1/2} - 1\}}{z} \right] , \qquad (1.4.3)$$

where $H_0 = \dfrac{\dot{a}(t_0)}{a(t_0)}$ is the present Hubble's rate and $q_0 = -\dfrac{\ddot{a}(t_0)}{a(t_0)H_0^2}$ is the deceleration parameter at t_0, being the present age of the universe.

(d) Relation between present density, age and Hubble's rate :

From Friedmann equation (1.2.14) with cosmological constant (Λ), it is obtained that

$$H^2 + \frac{k}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}(\rho^{(m)} + \rho^{(r)}) , \qquad (1.4.4)$$

where $\rho^{(m)}$ and $\rho^{(r)}$ are matter density and radiation dennsity.

At $t = t_0$, equation (1.4.4) reduces to

$$H_0^2 + \frac{k}{a_0^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}(\rho_0^{(m)} + \rho_0^{(r)}) . \qquad (1.4.5)$$

At time t, the critical density is defined as

$$\rho^{(c)} = \frac{3H^2}{8\pi G} . \qquad (1.4.6)$$

At the present time $t = t_0$, equation (1.4.6) reduces to

$$\begin{aligned}\rho_0^{(c)} &= \frac{3H_0^2}{8\pi G} \\ &= 1.88 \times 10^{-29} h^2 g/cm^3 \\ &= 5.43 \times 10^{-47} h^2 GeV^4 ,\end{aligned} \tag{1.4.7}$$

as $H_0 = 100\ h\ km/sec\ Mpc = 2.13 \times 10^{-42}$ h GeV . According to WMAP, Hubble's parameter $h = 0.68$ upto maximum likelihood .

From equation (1.4.5), cosmic sum rule is obtained as

$$\Omega_{(m)} + \Omega_{(r)} + \Omega_{(k)} + \Omega_{(\Lambda)} = 1 , \tag{1.4.8}$$

where

$$\Omega_{(m)} = \frac{\rho_0^{(m)}}{\rho_0^{(c)}} = \frac{8\pi G}{3H_0^2}\rho_0^{(m)} , \tag{1.4.9}$$

$$\Omega_{(r)} = \frac{\rho_0^{(r)}}{\rho_0^{(c)}} = \frac{8\pi G}{3H_0^2}\rho_0^{(r)} , \tag{1.4.10}$$

$$\Omega_{(k)} = -\frac{3k}{8\pi G\rho_0^{(c)} a_0^2} = -\frac{k}{H_0^2 a_0^2} , \tag{1.4.11}$$

$$\text{and } \ \Omega_{\Lambda} = \frac{\Lambda}{8\pi G\rho_0^{(c)}} = \frac{\Lambda}{3H_0^2} . \tag{1.4.12}$$

$\Omega_{(r)}$ in equation (1.4.8) is negligible. Since the present value of radiation density $\rho_0^{(r)} \simeq 4.5 \times 10^{-34} h^2 g/cm^3 = 5.89 \times 10^{-52} h^2 GeV^4$.

The equation (1.4.8) becomes

$$\Omega_{(m)} + \Omega_{(k)} + \Omega_{(\Lambda)} = 1 . \tag{1.4.13}$$

From equation (1.4.1), we obtain

$$H = H_0 \frac{y'}{y} , \tag{1.4.14}$$

by defining new time $\tau = H_0(t - t_0)$ and writing

$$y = \frac{a(t)}{a(t_0)} = (1 + z)^{-1}\,. \tag{1.4.15}$$

here y' means derivative with respect to τ.

The energy density for matter is obtained as

$$\rho^{(m)} = \rho_0^{(m)} \frac{a_0^3}{a^3}\,, \tag{1.4.16}$$

where present matter density, according to Wilkinson Cosmic Microwave Background Radiation Anisotropy Probe (WMAP) data is $\rho_0^{(m)} = 1.3 \times 10^{-30} g/cm^3$.

Using the matter density $\rho^{(m)}$, given by equation (1.4.4) is obtained as

$$(\frac{y\prime}{y})^2 - \frac{\Omega_{(k)}}{y^2} - \Omega_{(\Lambda)} = \frac{\Omega_{(m)}}{y^3}\,, \tag{1.4.17}$$

using equations (1.4.9) to (1.4.12) as $\Omega_{(r)} \approx 0$.

From equations (1.4.14) and (1.4.15) we obtain the initial conditions as

$$y(0) = 1 = y'(0).$$

Further connecting equations (1.4.13) and (1.4.17), we obtain

$$y' = [1 + (y^{-1} - 1)\Omega_{(m)} + (y^2 - 1)\Omega_\Lambda]^{1/2}\,. \tag{1.4.18}$$

From equation (1.4.18), it is obtained that

$$\int_{-H_0 t_0}^{0} dy = \int_0^1 \left[1 + (y^{-1} - 1)\Omega_{(m)} + (y^2 - 1)\Omega_\Lambda\right]^{-1/2} d\tau\,, \tag{1.4.19}$$

which yields

$$H_0 t_0 = \int_0^1 \left[1 + (y^{-1} - 1)\Omega_{(m)} + (y^2 - 1)\Omega_\Lambda\right]^{-1/2} d\tau\,. \tag{1.4.20}$$

Chapter 2

Standard Big-Bang Cosmology

The most successful scientific theory today about the origin and evolution of the universe is the Big-Bang theory, which is one of the most ambitious intellectual constructions of the huminity. It is based on two consolidated branches of theoretical physics, namely, the theory of General Relativity (GR) [20] and the Standard Model of Particle physics (SM), and is able to make robust predictions, such as the expansion of the universe, the existence and properties of the Cosmic Microwave Background Radiation (CMB) and the relative primordial abundance of light elements.

According to the standard Big-Bang theory, our universe emerged from a tremendous explosion, in which both space and time were created. The early universe was extremely hot, dense and rapidly expanding, while today the universe is cold - as suggested by the measured CMB temperature $T_{CMB} \simeq 2.73K$ - almost empty and it is still expanding. Thus, one can say that the history of the universe is the history of its expansion, which involves various important qualitative changes of its characteristics.

At a lower energy scale ($T \sim 10^2 GeV,\ t \sim 10^{-10} s$), the electro-weak symmetry is broken, and starting from that moment, the spectrum of particles in thermal equilibrium is that of the known particles produced in terrestrial accelerators. In this symmetry breaking, an asymmetry between matter and antimatter might also be generated, through a process known as electro-weak baryogenesis.

When the universe has a temperature of about $T \sim 1MeV(t \sim 1s)$, it contains neutrons, protons, electrons, positrons, neutrinos and photons. At that moment, fusion nuclear reactions may start and the lightest nuclei are formed $(H, He, Li, \dots)$. The processes of light nuclei formation occure out of equilibrium and they are known as Big-Bang (or primordial) Nucleosynthesis (BBN). Its predictions constitute a solid pillar for the Big-Bang theory and they can be corroborated by observations.

At temperature $T \sim 3eV(t \sim 10^{11} s)$ the energy density in non-relativistic matter becomes equal to that in relativistic particles, after which the universe becomes matter dominated. This is known as the epoch of matter-radiation equality, and is the epoch when structure formation becomes possible.

The next important moment in the history of the universe is when the nuclei, formed during Primordial Nucleosynthesis, combine with the existing electrons to form atoms. This occurs at a temperature $T \sim 0.3eV\,(t \sim 10^{13} s)$ and is known as recombination. It is the moment when the universe becomes transparent to light, because there are no more electrons to scatter the protons, which can now travel freely. Thus, the photons that reach us from that epoch constitute the oldest "picture" of the universe, which can be seen in the CMB radiation. After that, the recent history of the universe begins, and the processes that occur are the formation of stars, galaxies, planets and so on.

At present, observations of the universe indicate the existence of two unknown forms of energy, which dominate the energy content of the universe: dark matter and dark energy, which constitutes one of the most important problems of the modern cosmology.

2.1 The hot Big-Bang scenario

A priori, we don't know what type of fluid or particle gives the dominant contributions to the energy density of the universe. According to the Friedmann equation, this equation is related to many fundamental issues, like the behavior of the scale factor, the spatial curvature, or the past and future evolution of the universe.

2.1.1 Various possible scenarios for the history of the universe

We will classify the various types of matter that could fill the universe according to their pressure - to - density ratio. The three most likely possibilities are:

(i) Ultra-relativistic particles, with $v \simeq c$, $p \simeq \dfrac{\rho}{3}$, $\rho \propto a^{-4}$. This includes photons, massless neutrinos and eventually other particles that would have a very small mass and would be travelling at the speed of light. The generic name for this kind of matter, which propagates like electromagnetic radiation, is precisely "radiation".

(ii) Non-relativistic pressureless matter - in general, simply called "matter" by opposition to radiation with $v \ll c$, $p \simeq 0$, $\rho \propto a^{-3}$. This applies essentially to all structures in the universe: planets, stars, clouds of gas or galaxies seen as a whole.

(iii) A possible cosmological constant, with time invariant energy density and $p = -\rho$, that might be related to the vacuum of the theory describing elementary particles, or to something more mysterious. Whatever it is, we leave such a constant term as an open possibility. Following the definition given by Einstein, what is actually called the "cosmological constant" Λ is not the energy density ρ_{Λ} but the quantity

$$\Lambda = \frac{8\pi G \rho_{\Lambda}}{c^2}\,, \tag{2.1.1}$$

which has the dimension of the universe square of a time.

We write the Friedmann equation including these terms

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho_R + \frac{8\pi G}{3c^2}\rho_M - \frac{kc^2}{a^2} + \frac{\Lambda}{3}\,, \tag{2.1.2}$$

where ρ_R is the radiation density and ρ_M the matter density. The order in which we wrote the four terms on the right-hand side - radiation, matter, spatial curvature, cosmological constant - is not arbitrary. Indeed, they evolve with respect to the scale factor as a^{-4}, a^{-3}, a^{-2} and a^0. So, if the scale factors keep growing, and if these four terms are present in the universe, there is a chance that they all dominate the expansion of the universe one after each other. Of course, it is also possible that some of these terms donot exist at all, or are simply negligible. For instance, some possible scenarios would be:

(i) only matter domination, from the initial singularity until today.

(ii) radiation domination $\longrightarrow$ matter domination today.

(iii) radiation domination $\longrightarrow$ matter domination $\longrightarrow$ curvature domination today.

(iv) radiation domination $\longrightarrow$ matter domination $\longrightarrow$ cosmological constant domination today.

But all the cases that donot respect the order (like for instance: curvature domination $\longrightarrow$ matter domination) are impossible.

During each stage, one component strongly dominates the others, and the behavior of the scale factor, of the Hubble parameter and of the Hubble radius are given by

(a) Radiation domination:

$$\frac{\dot{a}^2}{a^2} \propto a^{-4}, \quad a(t) \propto t^{\frac{1}{2}}\,, \quad H(t) = \frac{1}{2t}, \quad R_H(t) = 2ct\,. \tag{2.1.3}$$

So, the universe is in decelerated power-law expansion.

(b) Matter domination:

$$\frac{\dot{a}^2}{a^2} \propto a^{-3}, \quad a(t) \propto t^{\frac{2}{3}}\,, \quad H(t) = \frac{2}{3t}, \quad R_H(t) = \frac{3}{2}ct\,. \tag{2.1.4}$$

Again, the universe is in power-law expansion, but it decelerates more slowly than during radiation domination.

(c) Negative curvature domination ($k < 0$):

$$\frac{\dot{a}^2}{a^2} \propto a^{-2}, \quad a(t) \propto t, \quad H(t) = \frac{1}{t}, \quad R_H(t) = ct. \tag{2.1.5}$$

An open universe dominated by its curvature is in linear expansion.

(d) Positive curvature domination:

If $k > 0$, and if there is no cosmological constant, the right-hand side finally goes to zero: expansion stops. After, the scale factor starts to decrese. H is negative, but the right-hand side of the Friedmann equation remains positive. The universe recollapses. We know that we are not in such a phase, because we observe the universe expansion. But a priori, we might be living in a closed universe, slightly before the expansion stops.

(e) Cosmological constant domination:

$$\frac{\dot{a}^2}{a^2} \propto \text{constant}, \quad a(t) \propto \exp(\frac{\Lambda t}{3}), \quad H = \frac{c}{R_H} = \sqrt{\frac{\Lambda}{3}}. \tag{2.1.6}$$

The universe ends up in exponentially accelerated expansion.

So, in all cases, there seems to be a time in the past at which the scale factor goes to zero, called the initial singularity or the “Big-Bang”. The Friedmann description of the universe is not supposed to hold until $a(t) = 0$. At some time, when the density reaches a critical value called the planck density, we believe that gravity has to be described by a quantum theory, where the classical motion of time and space disappears. Some proposals for such theories exist, mainly in the framework of "string theories". Sometimes, string theories try to address the initial singularity problem, and to bulid various scenarios for the origin of the universe. Anyway, this field is still very speculative and of course, our understanding of the origin of the universe will always break down at some point. A reasonable goal is just to go back as far as possible, on the basis of testable theories.

The future evolution of the universe heavily depends on the existence of a cosmological constant. If the later is exactly zero, then the future evolution is dictated by the curvature (if $k > 0$, universe will end up with a "Big Crunch", where quantum gravity will show up again and if $k \leq 0$ there will be eternal decelerated expansion). If instead there is a positive cosmological term which never decays into matter or radiation, then the universe necessarily ends up in eternal accelerated expansion.

2.1.2 The matter budget today

In order to know the past and future evolution of the universe, it would be enough to measure the present density of radiation, matter and Λ, and also to measure H_0. Again it would be possible to extrapolate $a(t)$ at any time from Friedmann equation. We take the Friedmann equation, evaluated today and divide it by H_0^2

$$1 = \frac{8\pi G}{3H_0^2c^2}\left(\rho_{_{R0}} + \rho_{_{M0}}\right) - \frac{kc^2}{a_0^2H_0^2} + \frac{\Lambda}{3H_0^2}\,, \tag{2.1.7}$$

where the subscript 0 means "evaluated today". Since by construction, the sum of these four terms is one, they represent the relative contributions to the present universe expansion. These terms are usually written as

$$\Omega_R = \frac{8\pi G}{3H_0^2c^2}\,\rho_{R0}, \tag{2.1.8}$$

$$\Omega_M = \frac{8\pi G}{3H_0^2c^2}\,\rho_{M0}, \tag{2.1.9}$$

$$\Omega_k = \frac{kc^2}{a_0^2H_0^2}, \tag{2.1.10}$$

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2}, \tag{2.1.11}$$

and the "matter budget" equation is

$$\Omega_R + \Omega_M - \Omega_k + \Omega_\Lambda = 1\,. \tag{2.1.12}$$

The universe is flat provided that

$$\Omega_0 \equiv \Omega_R + \Omega_M + \Omega_\Lambda\,, \tag{2.1.13}$$

is equal to one. In that case, as we already know, the total density of matter, radiation and Λ is equal at any time to the critical density,

$$\rho_c(t) = \frac{3c^2H^2(t)}{8\pi G}\,. \tag{2.1.14}$$

Note that the parameters Ω_x, where $x \in (R, M, \Lambda)$ could have been defined as the present density of each species divided by the present critical density,

$$\Omega_x = \frac{\rho_{x0}}{\rho_{c0}}. \tag{2.1.15}$$

So far, we conclude that the evolution of the Friedmann universe can be described entirely in terms of four parameters, called the "cosmological parameters": Ω_R, Ω_M, Ω_Λ, H_0. One of the main purposes of observational cosmology is to measure the value of these cosmological parameters.

2.2 Observational evidence for dark energy

A. Supernovae

In 1998 the accelerated expansion of the universe was pointed out by two groups (High-z Supernova Search Team [2] and in 1999 by the supernova cosmology project [1]) from the observations of Type Ia Supernove (SN Ia) by treating SN Ia as an ideal standard candle. Since then, these observations have been corroborated by several independent sources. Measurements of the cosmic microwave background, gravitational lensing, and the large scale structure of the cosmos as well as improved measurements of supernovae have been consistent with the Lambda-CDM model [21].

Supernovae are useful for cosmology because they are excellent standard candles across cosmological distances. They allow the expansion history of the universe to be measured by looking at the relationship between the distance to an object and its redshift, which gives how fast it is receding from us. The relationship is roughly linear, according to Hubble's law. It is relatively easy to measure redshift, but finding the distance to an object is more difficult. Usually, astronomers use standard candles: objects for which the intrinsic brightness, the absolute magnitude, is known. This allows the object's distance to be measured from its actually observed brightness, or apparent magnitude. Type Ia Supernovae are the best-known standard candles across cosmological distances because of their extreme, and extremely consistent brightness.

B. Cosmic microwave background

The case for an accelerating universe also received independent support from cosmic microwave background (CMB). The existence of dark energy, in whatever form, is needed to reconcile the measured geometry of space with the total amount of matter in the universe. Measurements of cosmic microwave background anisotropies, most recently by the WMAP satellite, indicate that the universe is very close to flat. For the shap of the universe to be flat, the mass-energy density of the universe must be equal to a certain critical density. The total amount of matter in the universe (including baryons and dark matter), as measured by the CMB, accounts for only about 30% of the critical density. This implies the existence of an additional form of energy to account for the remaining 70% [21].

C. Large-scale structure

The theory of large scale structure, which governs the formation of structure in the universe (stars, quasars, galaxies and galaxy clusters), also suggests that the density of baryonic matter in the universe is only 30% of the critical density.

D. Late-time integrated Sach's-Wolfe effect

Accelerated cosmic expansion causes gravitational potential wells and hills to flatten as photons pass through them, producing cold spots and hot spots on the CMB aligned with vast supervoids and superclusters. This so-called late-time integrated Sachs-Wolfe effect (ISW) is a direct signal of dark energy in a flat universe,[22] and has recently been detected at high significance by Ho et al. [23] and Giannantonio et al. [24]. In 2008, Granett, Neyrinck & Szapudi found arguably the clearest evidence yet for the ISW effect, imaging the average imprint of superclusters and supervoids on the CMB.

Apart from this, standard cosmological models also motivated for dark energy. Naturally a cosmological model is required to explain acceleration in the present universe. Such a model requires a source of energy to derive acceleration in the late universe. To search a suitable source of energy, it is important to know the nature of the perfect fluid of the required energy source. It is because, scale factor, being a solution of Friedmann equations, depends on the nature of dominating fluid.

Friedmann equations, with $\Lambda \neq 0$ are given by equations (1.2.14) and (1.2.17) as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}, \tag{2.2.1}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{2.2.2}$$

where $a(t)$ is the sclae factor of the universe, given by spatially flat FRW model

$$ds^2 = dt^2 - a^2(t)\left[dx^2 + dy^2 + dz^2\right], \tag{2.2.3}$$

supported by observations made by A.D. Miller et al. In equations (2.2.1) and (2.2.2), Λ has sign of negative pressure. It shows that repulsive nature of Λ giving anti-gravitational effect. Fluids satisfying $\rho + 3p \geq 0$ are said to satisfy the "strong energy condition" (SEC). From equation (2.2.2) we find that, in order to accelerate "dark energy" must violate strong energy condition (SEC).

2.3 Inflation and dark energy

The horizon and the flatness problems of the standard big bang cosmology are so serious that the theory seems to require some basic modifications of the hypothesis made so far. The most elegant solution is to suppose that the universe has gone through a non-adiabatic period and also through a period of accelerated expansion, during which physical scales Λ evolved much faster than the horizon scale H^{-1}. This period of positive acceleration, $\ddot{a} > 0$, of the primeval universe is called inflation.

The inflationary hypothesis is attractive because it holds out the possibility of calculating cosmological quantities, given the Lagrangian describing the fundamental interactions. In the context of the Standard Model, it is not possible to incorporate inflation, but this should not be regarded as a serious problem because the Standard Model itself requires modifications at higher energy scales, for reasons that have nothing to do with cosmology.

The negative active gravitational mass density associated with a positive cosmological constant is an early precursor of the inflation picture of the early universe, inflation in turn is one precursor of the idea that Λ might generalize into evolving dark energy.

To begin, we review some aspects of causal relations between events in spacetime. Neglecting space curvature, a light ray moves proper distance $dl =$

$a(t)dx = dt$ in time interval dt, so the integrated co-ordinate displacement is

$$x = \int \frac{dt}{a(t)}. \tag{2.3.1}$$

If $\Omega_{\Lambda 0} = 0$, this integral converges in the past- we see distant galaxies that at the time of observation cannot have seen us since the singular star of expansion at $a = 0$. This "particle horizon problem" is curious: how could distant galaxies in different directions in the sky know to look so similar? The inflation idea is that in the early universe the expansion history approximates that of de-sitter's (1917) solution to Einstein's field equation for $\Lambda > 0$ and $T_{\mu\nu} = 0$ in the field equation

$$G_{\mu\nu} = 8\pi G \left(T_{\mu\nu} + \rho\Lambda g_{\mu\nu}\right). \tag{2.3.2}$$

We can choose the co-ordinate labels in this de-sitter spacetime so space curvature vanishes. From equation (1.2.16) the energy density ρ is constant for $w = -1$. In this case the Hubble rate is also constant from equation (1.2.14), giving the evolution of the scale factor

$$a \propto e^{Ht}. \tag{2.3.3}$$

As one sees by working the integral in equation (2.3.1), here everyone can have seen everyone else in the past. For the following discussion two concepts are important. First, the early universe acts like an approximation to de sitter's solution because it is dominated by a large effective cosmological "constant", or dark energy density. Second, the dark energy is modeled as that of a near homogeneous field, ϕ.

In this scalar field model, motivated by grand unified models of very high energy particle physics, the action of the real scalar field, ϕ is (in units chosen so Planck's constant h is unity)

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] . \tag{2.3.4}$$

The potential energy density V is a function of the field ϕ, and g is the determinant of the metric tensor. When the field is spatially homogeneous and space curvature may be neglected, the field equation is

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{dV}{d\phi} = 0 . \tag{2.3.5}$$

The stress-energy tensor of this homogeneous field is diagonal (in the rest frame of an observer moving so the universe is seen to be isotropic), with time and space parts along the diagonal

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{2.3.6}$$

$$p_\phi = -\frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{2.3.7}$$

If the scalar field varies slowly in time, so that $\dot{\phi}^2 \ll V$, the field energy approximates the effect of Einstein's cosmological constant, with $p_\phi \simeq -\rho_\phi$.

The inflation picture assumes the near exponential expansion of equation (2.3.3) in the early universe lasts long enough that every bit of the present observable universe has seen every other bit, and presumably has discovered how to relax to almost exact homogeneity. The field ϕ may then start varying rapidly enough to produce the entropy of our universe, and the field or the entropy of our universe, and the field or the entropy may produce the baryons, leaving ρ_ϕ small or zero. But one can imagine the late time evolution of ρ_ϕ is slow. If slower than the evolution in the mass density in matter, there comes a time when ρ_ϕ again dominates, and the universe appears to have a cosmological constant.

A model for this late time evolution assumes a potential of the form

$$V = \frac{k}{\phi}a, \tag{2.3.8}$$

where the constant k has dimensions of mass raised to the power $\alpha + 4$. For simplicity let us suppose the universe after inflation but at high redshift is dominated by matter or radiation, with mass density ρ, that drives power law expansion, $a = t^n$, then the power law solution to the field equation, $a \propto t^n$. Then the power law solution to the field equation (2.3.5) with potential in equation (2.3.8) is

$$\phi \propto t^{\frac{2}{(2+\alpha)}}, \tag{2.3.9}$$

and the ratio of the mass densities in the scalar field and in matter or radiation is

$$\frac{\rho_\phi}{\rho_\alpha} t^{\frac{4}{(2+\alpha)}}. \tag{2.3.10}$$

In the limit where the parameter α approaches zero, ρ_ϕ is constant and this model is equivalent to Einstein's Λ.

When $\alpha > 0$ the field ϕ in this model grows arbitrarily large at large time, so $\rho_\phi \longrightarrow 0$, and the universe approaches the Minkowskian spacetime of special relativity. This is within a simple model, ofcourse. It is easy to imagine that in other models ρ_ϕ approaches a constant positive value at large time and space-time approaches the de-sitter solution, or ρ_ϕ passes through zero and becomes negative, causing space-time to collapse to a Big Crunch.

The power law model with $\alpha > 0$ has two properties that seem desirable. First, the solution in equation (2.3.9) is said to be an attractor (Ratra and Peebless, 1998) or tracker (Steinhardt, Wang, and Zlatev, 1999), meaning it is the asymptotic solution for a broad range of initial conditions at high redshift. That includes relaxation to a near homogeneous energy distribution even when gravity has collected the other matter into non-relativistic clumps. Second, the energy density in the attractor solution decreases less rapidly than that of matter and radiation. This allows us to realize the scenario: after inflation but at high redshift the field energy density ρ_ϕ is small so it does not disturb the standard model for the origin of the light elements, but eventually ρ_ϕ dominates and the universe acts as if it had a cosmological constant, but one that varies slowly with position and time.

2.4 A brief developments of dark energy model

The discovery in 1998 that the universe is actually speeding up its expansion was a total shock to astronomers. This result revolutionized cosmologists thinking about the present universe. Naturally, question arises what is the source of energy that derive acceleration in the universe. So firstly it was thought that cosmological constant Λ could be driving cosmic acceleration in the late universe too as it is responsible for acceleration in the early universe (inflation). But this proposal faced a problem that, in the beginning of the universe, it has a very high value. For example at Plank scale it is $\sim 10^{75} GeV^4$ and at Quantum Chromodynamics scale (QCD) it has $\sim 10^{-6} GeV^4$. But in the present universe, dark energy density is found to be around $2 \times 10^{-47} GeV^4$. To satisfy these extreme values of dark energy, it was proposed that a dynamical model of Λ should be introduced.

Modern cosmology heavily depends on particle physics. Role of scalar fields is very important in particle physics, so it is natural to think of scalar field being a source of dark energy. Prompted by this idea, cosmologists proposed scalar field models of dark energy in different forms. Among these scalar field models, quintessence model [10, 25], K-essence model [11], tachyon model [13], phantom model [14] and dilatonic models are important. Contributions from scalar fields, that are constant in space are usually also included in cosmological constant. The cosmological constant is physically equivalent to vacuum energy [7]. Scalar fields which do change in space can be difficult to distinguish from a cosmological constant because the change may be extremly slow. Apart from scalar field models, Chaplygin gas model [16] (having negative pressure) has been worked out. Moreover, in the past few years, it is probed whether, curvature of the space-time manifold, describing the universe, could provide dark energy in its own right.

Dark energy models mainly rely on the implicit assumption that Einstein's General Relativity is the correct theory of gravity indeed. Neverthless, its validity on large astrophysical and cosmological scales has never been tested, but only assumed and it is therefore conceivable that both cosmic speed up and missing matter are nothing else but signals of a break down of

General Relativity.

The aim of dark energy models is to explain the accelerated expansion of the universe. Like for inflationary models producing an early stage of a accelerated expansion, we have how a wide variety of dark energy models that can account for the late time background evolution. In the same way that inflationary models are constrained by the cosmological perturbation. They produce, dark energy models can be constrained by the background evolution and their effect on the growth of perturbations. As for all dark energy models, scalar-tensor dark energy models are characterized by the accelerated expansion they produce at low redshifts, but this background effect is common to all dark energy models.

Chapter 3

Dark Energy and the Cosmological Constant

3.1 Introduction

The cosmological constant Λ, was originally introduced by Einstein in 1917 to achieve a static universe. After Hubble's discovery of the expansion of the universe in 1929, it was dropped by Einstein as it was no longer required. From the point of view of particle physics, however, the cosmological constant naturally arieses as an energy density of the vacuum. More-over, the energy scale of Λ should be much larger than that of the present Hubble constant H_0, if it originates from the vacuum energy density. This is the "cosmological constant problem" [26] and was well known to exist long before the discovery of the accelerated expansion of the universe in 1998.

Recent years have witnessed a resurgence of interest in the possibility that a cosmological constant Λ may dominate the total energy density in the universe. Interest in the cosmological constant stems from several directions:

(i) Observations of high redshift Type Ia Supernovae appear to suggest that our universe may be accelerating with a large fraction of the cosmological density in the form of a cosmological Λ-term. When combined with observations of the cosmic microwave background (CMB), an approximately flat Friedmann-Robertson-Walker (FRW) cosmological model with total energy density ($\Omega_m + \Omega_\Lambda \simeq 1$) is suggested, in agreement with predictions of the simplest versions of the inflationary scenario of the early universe.

(ii) Most dynamical estimates of the amount of clustered matter yield a conservative upper limit $\Omega_m \lesssim 0.3$. In addition, theoretical modelling of structure formation based on the cold dark matter model (CDM) with $\Omega = 1$ has failed to match up with observations at a quantitative level. By contrast, a flat low density CDM+Λ universe with $\Omega_m \simeq 0.3$ and $\Omega_\Lambda \simeq 0.7$, and with an approximately flat (or, Harriso-Zeldovich-like, $n_s \approx 1$) initial Fourier spectrum of scalar (adiabatic) inhomogeneous metric and density perturbations agrees remarkably well with a wide range of observational data ranging from large and intermediate angle CMB anisotropies to observations of galaxy clustering on large scales. Since an approximately flat initial spectrum of adiabatic perturbations is also precisely what simplest variants of the inflationary scenario predict, the positive Λ-term removes a necessity in any complications of the inflationary scenario (which might be required if the universe was found to be open).

(iii) At a theoretical level, a cosmological constant $\Lambda = 8\pi G\rho_{vac}/c^2$ is predicted to arise out of zero-point quantum vacuum fluctuations of fundamental scalar, spinor, vector and tesor fields. Although a theoretically predicted value of ρ_{vac} usually appears to be much larger than current observational limits, there is no generic known mechanism which will set the value of Λ to precisely zero either on the basis of symmetry arguments or by dynamical means.

3.2 Cosmological constant Λ

The Einstein tensor $G^{\mu\nu}$ and the energy momentum tensor $T^{\mu\nu}$ satisfy the Bianchi identities $\bigtriangledown_\nu G^{\mu\nu} = 0$ and energy conservation $\bigtriangledown_\nu T^{\mu\nu} = 0$. Since the metric $g^{\mu\nu}$ is constant with respect to covariant derivatives ($\bigtriangledown_\alpha g^{\mu\nu} = 0$), there is a freedom to add a term $\Lambda g_{\mu\nu}$ in the Einstein equations. Then the modified Einstein-equations are given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}\,. \tag{3.2.1}$$

By taking a trace of this equation, we find that $-R + 4\pi = 8\pi GT$. Combining this relation with equation (3.2.1), we obtain

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})\,. \tag{3.2.2}$$

Let us consider Newtonian gravity with metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is the perturbation around the Minkowski metric $\eta_{\mu\nu}$. If we neglect the time-variation and rotational effect of the metric, R_{00} can be written by a gravitational potential ϕ, as $R_{00} \simeq -(\frac{1}{2})\Delta h_{00} = \Delta\phi$. Here g_{00} is given by $g_{00} = -1 - 2\phi$. In the relativistic limit with $|p| \ll \rho$, we have $T_{00} \simeq -T \simeq \rho$. then the 00 component of equation (3.2.2) gives

$$\Delta\phi = 4\pi G\rho - \Lambda\,. \tag{3.2.3}$$

In order to reproduce the Poisson equation in Newtonian gravity, we require that $\Lambda = 0$ or Λ is sufficiently small relative to the $4\pi G\rho$ term in equation (3.2.3). Since Λ has dimensions of $[\text{Length}]^{-2}$, the scale corresponding to the cosmological constant needs to be much larger than the scale of stellar objects on which Newtonian gravity works well. In other words the cosmological constant becomes important on very large scales.

In the FRW background given by equation (1.2.1), the modified Einstein equations (3.2.1) give

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3}, \tag{3.2.4}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{3.2.5}$$

This clearly demonstrtes that the cosmological constant contributes negaively to the pressure term and hence exhibits a repulsive effect.

Let us consider a static universe (a=constant) in the absence of Λ. Setting $H = 0$ and $\frac{\ddot{a}}{a} = 0$ in equations (1.2.14) and (1.2.17), we find

$$\rho = -3p = \frac{3K}{8\pi G a^2}. \tag{3.2.6}$$

Equation (3.2.6) shows that either ρ or p needs to be negative. When Einstein first tried to construct a static universe, he considered that the above solution is not physical and so added the cosmological constant to the original field equations (1.2.9).

Using the modified field equations (3.2.4) and (3.2.5) in a dust-dominated universe ($p = 0$), we find that the static universe obtained by Einstein corresponds to

$$\rho = \frac{\Lambda}{4\pi G}\,, \qquad \text{where } \Lambda = \frac{K}{a^2}\,. \tag{3.2.7}$$

Since $\rho > 0$ we require that Λ is positive. This means that the static universe is a closed one ($K = +1$) with a radius a $a = \frac{1}{\sqrt{\Lambda}}$. Equation (3.2.7) shows that the energy density ρ is determined by Λ.

The requirement of a cosmological constant to achieve a static universe can be understood by having a look at the Newton's equation of motion in equation (1.2.17). Since gravity pulls the point particle toward the centre of the sphere, we need a repulsive force to realize a situation in which a is constant. This corresponds to adding a cosmological constant term $\frac{\Lambda}{3}$ on the right hand side of equation (1.2.17).

The above description of the static universe was abandoned with the discovery of the redshift of distant stars, but it is intriguing that such a cosmological constant should return in the 1990's to explain the observed acceleration of the universe.

Introducing the modified energy density and pressure

$$\tilde{\rho} = \rho + \frac{\Lambda}{8\pi G}, \quad \tilde{p} = p - \frac{\Lambda}{8\pi G}\,, \tag{3.2.8}$$

we find that equations (3.2.4) and (3.2.5) reduce to equations (1.2.14) and (1.2.17).

Fine tuning problem of cosmological constant :

If the cosmological constant originates from a vacuum energy density, then this suffers from a severe fine-tuning problem. Observationally we know that Λ is of order the present value of the Hubble parameter H_0, that is

$$\Lambda \approx H_0^2 = (2.13h \times 10^{-42} GeV)^2. \tag{3.2.9}$$

This corresponds to a critical density ρ_Λ,

$$\rho_\Lambda = \frac{\Lambda m_{pl}^2}{8\pi} \approx 10^{-47} GeV^4. \tag{3.2.10}$$

Meanwhile the vacuum energy density evaluated by the sum of zero-point energies of quantum fields with mass m is given by

$$\begin{aligned} \rho_{vac} &= \frac{1}{2}\int_0^\infty \frac{d^3k}{(2\pi)^3}\sqrt{k^2+m^2}\,, \\ &= \frac{1}{4\pi^2}\int_0^\infty dk k^2\sqrt{k^2+m^2}\,. \end{aligned} \tag{3.2.11}$$

This exhibits an ultraviolet divergence : $\rho_{vac} \propto k^4$. However we expect that quantum field theory is valid up to some cut-off scale k_{max} in which case the integral given by equation (3.2.11) is finite,

$$\rho_{vac} \approx \frac{k_{max}^4}{16\pi^2}. \tag{3.2.12}$$

For the extreme case of General Relativity we expect it to be valid to just below the Plank scale : $m_{pl} = 1.22 \times 10^{19}$GeV. Hence if we pick up $k_{max} = m_{pl}$, we find that the vacuum energy density in this case is estimated as

$$\rho_{vac} \approx 10^{74} GeV^4, \tag{3.2.13}$$

which is about 10^{121} orders of magnitude larger than the observed value given by equation (3.2.10). Even if we take an energy scale of QCD for k_{max}, we obtain $\rho_{vac} \approx 10^{-3} GeV^4$, which is still much larger than ρ_Λ.

We note that this contribution is related to the ordering ambiguity of fields and disappears when normal ordering is adopted. Since this procedure of throwing away the vacuum energy is adhoc, one may try to cancel it by introducing counter terms. However this requires a fine-tuning to adjust ρ_Λ to the present energy density of the universe. Whether or not the zero point energy in field theory is realistic is still a debatable question.

A nice resolution of the zero point energy is provided by supersymmetry. In supersymmetric theories every bosonic degree of freedom has its fermi counter part which contributes to the zero point energy with an opposite sign compared to the bosonic degree of freedom thereby canceling the vacuum energy. Indeed, for a field with spin $j > 0$, the expression given by equation (3.2.11) for the vaccum energy generalizes to

$$\begin{aligned} \rho_{vac} &= \frac{1}{2}(-1)^{2j}(2j+1)\int_0^\infty \frac{d^3k}{(2\pi)^3}\sqrt{k^2+m^2}\,, \\ &= \frac{(-1)^{2j}(2j+1)}{4\pi^2}\int_0^\infty dk\, k^2\sqrt{k^2+m^2}\,. \end{aligned} \tag{3.2.14}$$

Exact sepersymmetry (SUSY) implies an equal number of fermionic and bosonic degrees of freedom for a given value of the mass m such that the net contribution to the vaccum energy vanishes. It is in this sense that supersymmetric theories do not admit a non-zero cosmological constant. Hoeever, we know that we do not live in a supersymmetric vacuum state and hence it should be broken today. For a viable supersymmetric scenario, for instance if it is to be relevant to the hierarchy problem, the supersymmetry breaking scale should be around $M_{SUSY} \sim 10^3$GeV. Indeed, the presence of a scalar field (Higgs field) in the standard model of particle physics is necessary to ensure the possibility of a spontaneous breakdown of the gauge symmetry. However, the same scalar field creates what has come to be known as the "hierarchy problem". The origin of this problem lies in the quadratic nature of the divergence of the scalar self-energy arising out of scalar loops. A way out of this is supersymmetry (SUSY) which as we have mentioned demands a fermionic partner for every boson and vice versa with the two having the same mass. Since fermionic loops come with an over all negativesign, the divergence in the scalar self energy due to the scalar loop and its SUSY partner cancel out. However, particles in nature donot come with degenerate partners as demanded by SUSY and hence SUSY must be broken.

With supersymmetry breaking around 10^3GeV, we are still far away from the observed value of Λ by many orders of magnitudes. At present we donot know how the Plank scale or SUSY breaking scales are really related to the observed vacuum scale.

3.3 The many faces of the cosmoligical constant

Einstein's equations, which determine the dynamics of the spacetime, can be derived from the action

$$A = \frac{1}{16\pi G}\int R\sqrt{-g}d^4x + \int L_{matter}(\phi, \partial\phi)\sqrt{-g}d^4x\,, \tag{3.3.1}$$

where L_{matter} is the Lagrangian for matter depending on some dynamical variables generically denoted as ϕ. (We are using units with $c = 1$). The variation of this action with respect to ϕ will lead to the equation of motion for matter $(\delta L_{matter}/\delta\phi) = 0$, in a given background geometry, while the variation of the action with respect to the metric tensor g_{ik} leads to the Einstein's equation

$$R_{ik} - \frac{1}{2}g_{ik}R = 16\pi G\frac{\delta L_{matter}}{\delta g^{ik}} \equiv 8\pi G T_{ik}. \tag{3.3.2}$$

where the last equation defines the energy momentum tensor of matter to be $T_{ik} \equiv 2(\delta L_{matter}/\delta g^{ik})$.

Now we consider a new matter action $L'_{matter} - (\Lambda/8\pi G)$, where Λ is a real constant. Equation of motion for the matter $(\delta L_{matter}/\delta\phi) = 0$, does not change under this transformation since Λ is a constant; but the action now picks up an extra term proportional to Λ,

$$\begin{aligned} A =& \frac{1}{16\pi G}\int R\sqrt{-g}d^4x + \int (L_{matter} - \frac{\Lambda}{8\pi G})\sqrt{-g}d^4x \\ =& \frac{1}{16\pi G}\int (R - 2\Lambda)\sqrt{-g}d^4x + \int L_{matter}\sqrt{-g}d^4x\,, \end{aligned} \tag{3.3.3}$$

and equation (3.3.2) gets modified. The innocuous looking addition of a constant to the matter Lagrangian leads to one of the most fundamental and fascinating problems of theoretical physics. The nature of this problem and its theoretical backdrop acquires different shades of meaning depending which of the two forms of equations in (3.3.3) is used.

The first interpretation, based on the first line of equation (3.3.3), treats Λ as the shift in the matter Lagrangian which, in turn, will lead to a shift in the matter Hamiltonian. This could be thought of as a shift in the zero point energy of the matter system. Such a constant shift in the energy does not affect the dynamics of matter while gravity - which couples to the total energy of the system - picks up an extra contribution in the form of a new term Q_{ik} in the energy-momentum tensor, leading to

$$R^i_k - \frac{1}{2}\delta^i_k R = 8\pi G(T^i_k + Q^i_k)\,;$$

where

$$Q^i_k \equiv \frac{\Lambda}{8\pi G}\delta^i_k \equiv \rho_\Lambda \delta^i_k. \tag{3.3.4}$$

The second line in equation (3.3.3) can be interpreted as gravitational field, described by the Lagrangian of the form $L_{grav} \propto \frac{1}{G}(R-2\pi)$, interacting with matter described by the Lagrangian L_{matter}. In this interpretation, gravity is described by two constants, the Newton's constant G and the cosmological constant Λ. It is then natural to modify the left hand side of Einstein's equations and write equation (3.3.4) as

$$R^i_k - \frac{1}{2}\delta^i_k R - \delta^i_k \Lambda = 8\pi G T^i_k\,. \tag{3.3.5}$$

In this interpretation, the spacetime is treated as curved even in the absence of matter ($T_{ik} = 0$), since the equation $R_{ik} - \frac{1}{2}g_{ik}R - \Lambda g_{ik} = 0$ does not admit flat spacetime as a solution. (This situation is rather unusual and is related to the fact that symmetries of the theory with and without a cosmological constant are drastically different; the original symmetry of general covariance cannot be naturally broken in such a way as to preserve the sub group of spacetime translations).

In fact, it is possible to consider a situation in which both effects can occur. If the gravitational interaction is actually described by the Lagrangian of the form $(R-2\Lambda)$ then there is an intrinsic cosmological constant in nature just as there is a Newtonian gravitational constant in nature just as there is a Newtonian gravitational constant in nature. If the matter Lagrangian contains energy densities which change due to dynamics, then L_{matter} can pick up constant shifts during dynamical evolution. For example, consider a scalar field with the Lagrangian $L_{matter} = \frac{1}{2}\partial_i\phi\partial^i\phi - V(\phi)$ which has the energy momentum tensor

$$T^a_b = \partial^a\phi\partial_b\phi - \delta^a_b(\frac{1}{2}\partial^i\phi\partial_i\phi - V(\phi)). \tag{3.3.6}$$

For field configurations which are constant [for example, at the minima of the potential $V(\phi)$], this contributes an energy momentum tensor $T^a_b = \delta^a_b V(\phi_{min})$ which has exactly the same form as a cosmological constant. As far as gravity is concerned, it is the combination of these two effects - of very different nature - which is relevant and the source will be $T^{eff}_{ab} = [V(\phi_{min}) + (\Lambda/8\pi G)]g_{ab}$, corresponding to an effective gravitational constant

$$\Lambda_{eff} = \Lambda + 8\pi G V(\phi_{min})\,. \tag{3.3.7}$$

If ϕ_{min} and hence $V(\phi_{min})$ changes during dynamical evolution, the value of Λ_{eff} can also change in course of time. More generally, any field configuration which is varying slowly in time will lead to a slowly varying Λ_{eff}.

The extra term Q_{ik} in Einstein's equation behaves in a manner which is very peculiar compared to the enerrgy momentum tensor of normal matter. The term $Q^i_k = \rho_\Lambda \delta^i_k$ is in the form of the energy momentum tensor of an ideal fluid with energy momentum tensor of an ideal fluid with energy density of this 'fluid' must be negative, which is unlike conventional laboratory systems [27].

Such an equation of state $\rho = -p$ also has another important implication in general relativity. The spatial part $\mathbf{g}$ of the geodesic acceleration (which measures the relative acceleration of two geodesics in the spacetime) satisfies the following exact equation in general relativity

$$\nabla . g = -4\pi G(\rho + 3p)\,, \tag{3.3.8}$$

showing that the source of geodesic acceleration is $(\rho + 3p)$ and not ρ. As long as $(\rho + 3p) > 0$, gravity remains attractive while $(\rho + 3p) < 0$ can lead to repulsive gravitational effects. Since the cosmological constant has $(\rho_\Lambda + 3p_\Lambda) = -2\rho_\Lambda$, a positive cosmological constant (with $\Lambda > 0$) can lead to repulsive gravity. For example, if the energy density of normal, non-relativistic matter with zero pressure is ρ_{NR}, then equation (3.3.8) shows that the geodesics will accelerate away from each other due to the repulsion of cosmological constant when $\rho_{NR} < 2\rho_\Lambda$. A related feature, which makes the above conclusion practically relevant is the fact that, in an expanding universe, ρ_Λ remains constant while ρ_{NR} decreases. (more formally, the equation of motion $d(\rho_\Lambda V) = -p_\Lambda dV$ for the cosmological constant, treated as an ideal fluid, is identically satisfied with constant $\rho_\Lambda,\ p_\Lambda$). Therefore, ρ_Λ will eventually dominate over ρ_{NR} if the universe expands sufficiently. Since $|\Lambda|^{1/2}$ has the dimensions of inverse length, it will set the scale for the universe when cosmological constant dominates.

It follows that the most stringent bounds on Λ will arise from cosmology when the expansion of the universe has diluted the matter energy density sufficiently. The rate of expansion of the universe today is usually expressed in terms of the Hubble constant $H_0 = 100hKmS^{-1}Mpc^{-1}$ where $1Mpc \approx$

$3 \times 10^{24} cm$ and h is a dimensionless parameter in the range $0.62 \lesssim h \lesssim 0.82$. From H_0 we can form the time scale $t_{univ} \equiv H_0^{-1} \approx 10^{10} h^{-1} yr$ and the length scale $cH_0^{-1} \approx 3000 h^{-1} Mpc$; t_{univ} characterizes the evolutionary time scale of the universe and H_0^{-1} is of the order of the largest length scales currently accessible in cosmological observations. From the observation that the universe is at least of the size H_0^{-1}, we can set a bound on Λ to be $|\Lambda| < 10^{-56} cm^{-2}$. This stringent bound leads to several issues which have been debated for decades without satisfactory solutions.

(a) In classical general relativity, based on the constants G, c and Λ, it is not possible to construct any dimensionless combination from these constants. Neverthless, it is clear that Λ is extremely tiny compared to any other physical scale in the universe, suggesting that Λ is probably zero. We however, do not know of any symmetry mechanism or invariance principle which requires Λ to vanish. Supersymmetry does require the vanishing of the ground state energy; however, supersymmetry is so badly broken in nature that this is not of any practical use [28].

(b) We mentioned above that observations actually constrain Λ_{eff} in equation (3.3.7), rather than Λ. This requires Λ and $V(\phi_{min})$ to be fine tuned to an enormous accuracy for the bound, $|\Lambda_{eff}| < 10^{-56} cm^{-2}$, to be satisfied. This becomes more mysterious when we realize that $V(\phi_{min})$ itself could change by several orders of magnitude during the evolution of the universe.

(c) When quantum fields in a given curved spacetime are considered (even without invoking any quantum gravitational effects) one introduces the

Planck constant, h in the description of the physical system. It is then possible to form the dimensionless combination $\Lambda(Gh/c^3) \equiv \Lambda L_p^2$. (This equation also defines the quantity L_p^2). The bound on Λ translates into the condition $\Lambda L_p^2 \lesssim 10^{-123}$. As it has been mentioned several times in literature, this will require enormous fine tuning.

(d) All the above questions could have been satisfactorily answered if we take Λ_{eff} to be zero and assume that the correct theory of quantum gravity will provide an explanation for the vanishing of cosmological constant. Current cosmological observations however suggests that Λ_{eff} is actually non zero and $\Lambda_{eff} L_P^2$ is indeed of order $\mathcal{O}(10^{-123})$. In some sense, this is the cosmologist's worst nightmare come true. If the observations are correct, then Λ_{eff} is non zero, very tiny and its value is extremely fine tuned for no good reason. This is the concrete statement of the first of the two 'cosmological constant problems'.

(e) The bound on ΛL_p^2 arises from the expansion rate of the universe or - equivalently - from the energy density which is present in the universe today. The observations require the energy density of normal, non relativistic matter to be of the same order of magnitude as the energy density contributed by the cosmological constant. But in the past, when the universe was smaller, the energy density of normal matter would have been higher while the energy density of cosmological constant does not change. Hence we need to adjust the energy densities of normal matter and cosmological constant in the early epoch very carefully so that $\rho_\Lambda \gtrsim \rho_m$ around the current epoch. If this had happened very early in the evolution of the universe, then the repulsive nature of a positive cosmological constant would have initiated a rapid expansion of the universe, preventing the formation of galaxies, stars etc. If the epoch of $\rho_\Lambda \approx \rho_m$ occurs much later in the future, then the current observations would not have revealed the presence of non zero cosmological constant. This raises the second of the two cosmological constant problems : Why is it that $(\rho_\Lambda/\rho_m) = \mathcal{O}(1)$ at the current phase of the universe ?

(f) The sign of Λ determines the nature of solutions to Einstein's equation as well as the sign of $(\rho_\Lambda + 3p_\Lambda)$. Hence the spacetime geometry with $\Lambda L_p^2 = 10^{-123}$ is very different from the one with $\Lambda L_p^2 = -10^{-123}$. Any theoretical principle which explains the near zero value of ΛL_p^2 must also explain why the observed value of Λ is positive.

3.4 Geometrical features of a universe with a cosmological constant

The evolution of the universe has different characteristic features if there exists sources in the universe for which $(1 + 3w) < 0$. This is obvious from equation (3.3.8) which shows that if $(\rho + 3p) = (1 + 3w)\rho$ becomes negative then the gravitational force of such a source (with $\rho > 0$) will be repulsive. The simplest example of this kind of a source is the cosmological constant with $w_\Lambda = -1$.

To see the effect of a cosmological constant we consider a universe with matter, radiation and a cosmological constant. Introducing a dimensionless time co-ordinate $\tau = H_0 t$ and writing $a = a_0 q(\tau)$, equation (1.2.14) can be cast in a more suggestive from describing the one dimensional motion of a particle in a potential

$$\frac{1}{2}(\frac{dq}{d\tau})^2 + V(q) = E\,, \tag{3.4.1}$$

where

$$V(q) = -\frac{1}{2}\left[\frac{\Omega_r}{q^2} + \frac{\Omega_m}{q} + \Omega_\Lambda q^2\right]; \quad E = \frac{1}{2}(1-\Omega)\,. \tag{3.4.2}$$

This equation has the structure of the first integral for motion of a particle with energy E in a potential $V(q)$. For models with $\Omega = \Omega_r + \Omega_m + \Omega_\Lambda = 1$, we can take $E = 0$ so that

$$(\frac{dq}{d\tau}) = \sqrt{V(q)}\,. \tag{3.4.3}$$

At redshift (small q) the universe is radiation dominated and $\dot{q}$ is independent of the other cosmological parameters. At low redshifts, the presence of cosmological constant makes a difference and infact - decreasing function to an increasing function. In other words, the presence of a accelerating universe at low redshifts.

For a universe with non relativistic matter and cosmological constant, the potential in equation (3.4.2) has a simple form, varying as $(-a^{-1})$ for small a and $(-a^2)$ for large a with a maximum in between at $q = q_{max} = (\Omega_m/2\Omega_\Lambda)^{1/3}$. This system has been analyzed in detail in literature for both constant, cosmological constant [29] and for a time dependent cosmological constant [30]. A wide variety of explicit solutions for $a(t)$ can be provided for these equations. We briefly summarize a few of them.

(a) If the 'particle' is situated at the top of the potential, one can obtain a static solution with $\ddot{a} = \dot{a} = 0$ by adjusting the cosmological constant and the dust energy density and taking $k = 1$. This solution,

$$\Lambda_{crit} = 4\pi G\rho_m = \frac{1}{a_0^2}\,, \tag{3.4.4}$$

was the one which originally prompted Einstein to introduce the cosmological constant.

(b) The above solution is, obviously, unstable and any small deviation from the equilibrium position will cause $a \longrightarrow 0$ or $a \longrightarrow \infty$. By fine tuning the values, it is possible to obtain a model for the universe which 'loiters' around $a = a_{max}$ for a large period of time [31].

(c) A subset of models corresponds to those without matter and driven entirely by cosmological constant Λ. These models have $k = (-1, 0, +1)$ and the corresponding expansion factors being propotional to $[\sinh(Ht), exp(Ht), \cosh(Ht)]$ with $\Lambda^2 = 3H^2$. These line elements represent three different characterizations of the de-sitter spacetime. The manifold is a four dimensional hyperboloid embeded in a flat, five dimensional space with signature (+ - - -).

(d) It is also possible to obtain solutions in which the particle starts from $a = 0$ with an energy which is insufficient for it to overcome the potential barrier. These models lead to a universe which collapses back to a singularity. By arranging the parameters suitably, it is possible to make $a(t)$ move away or towards the peak of the potential (which corresponds to the static Einstein universe asymptotically [29].

(e) In the case of $\Omega_m + \Omega_\Lambda = 1$, the explicit solution for $a(t)$ is given by

$$a(t) \propto (\sinh \frac{3}{2} H t)^{\frac{2}{3}}; \quad k = 0; \quad 3H^2 = \Lambda. \tag{3.4.5}$$

This solution smoothly interpolates between a matter dominated universe $a(t) \propto exp(Ht)$ at late stages. The transition from deceleration to acceleration occurs at $Z_{acc} = (2\Omega_\Lambda/\Omega_m)^{\frac{1}{3}} - 1$, while the energy densities of the cosmological constant and the matter are equal at $Z_{\Lambda m} = (\Omega_\Lambda/\Omega_m)^{\frac{1}{3}} - 1$.

3.5 Evidence for a non-zero cosmological constant

There are several cosmological observations which suggests the existence of a non zero cosmological constant with different levels of reliability. most of these determine either the value of Ω_m or some combination of Ω_m and Ω_Λ. When combined with the strong evidence from the CMBR observations that the $\Omega_{tot} = 1$ or some other independent estimate of Ω_m, one is led to a non zero value for Ω_Λ. The most reliable ones seem to be those based on high redshift supernova [1, 2] and structure formation models [32]. We shall now discuss some of these evidence.

3.5.1 Supernovae and the accelerating universe.

The evolution of a universe with $\Omega_\Lambda \neq 0$ changes from a decelerating phase to an accelerating phase at late times. If $H(a)$ can be observationally determined, then one can check whether the real universe had undergone an accelerating phase in the past. This, in turn, can be done if $d_L(z)$, say, can be observationally determined for a class of sources. Bright supernovae explosions are brief explosive stellar events which are broadly classified as two types. Type-Ia supernovae occurs when a degenerate dwarf star containing CNO enters a stage of rapid nuclear burning cooking iron group elements. These are the brightest and most homogeneous class of supernovae with hydrogen poor spectra. An empirical correlation has been observed between the sharply rising light curve in the initial phase of the supernovae and its peak luminosity so that they can serve as standard candles. These events typically last about a month occurs approximately once in 300 years in our galaxy. (Type II supernovae, which occur at the end of stellar evolution, are not useful as standard candles).

For any supernovae, one can directly observe the apparent magnitude m [which is essentially the logarithm of the flux F observed] and its redshift. The absolute magnitude M of the supernovae is again related to the actual luminosity L of the supernovae in a logarithmic fashion. Hence the relation $F = (L/4\pi d_L^2)$ can be written as

$$m - M = 5log_{10}(\frac{d_L}{Mpc}) + 25\,. \tag{3.5.1}$$

The numerical factors arise from the astronomical conventions used in the definition of m and M. Very often, one will use the dimensionless combination $(H_0 d_L(z)/c)$ rather than $d_L(z)$ and the above equation will change to $m(z) = M + 5log_{10}(H_0 d_L(z)/c)$ with the quantity μ being related to M by

$$\mu = M + 25 + 5log_{10}(\frac{cH_0^{-1}}{Mpc}) = M - 5log_{10}h + 42.38\,. \tag{3.5.2}$$

If the absolute magnitude of a class of Type I supernovae can be determined from the study of its light curve, then one can obtain the d_L for these supernovae at different redshifts. (In fact, we only need the low-z apparent magnitude at a given z which is equivalent to knowing μ). Knowing d_L, one can determine the geometry of the universe.

To understand this effect in a simple context, let us compare the luminosity distance for a matter dominated model ($\Omega_m = 1, \Omega_\Lambda = 0$)

$$d_L = 2H_0^{-1}[(1+z) - (1+z)^{\frac{1}{2}}]\,, \tag{3.5.3}$$

with that for a model driven purely by a cosmological constant ($\Omega_m = 0,\ \Omega_\Lambda = 1$)

$$d_L = h_0^{-1} z(1+z)\,. \tag{3.5.4}$$

It is clear that at a given z, the d_L is larger for the cosmological constant model. Hence, a given object, located at a fixed redshift, will appear brighter in the matter dominated model compared to the cosmological constant dominated model. Though this sounds easy in principle, the statistical analysis turns out to be quite complicated.

The supernovae cosmology project (SCP) has discovered [1] 42 supernovae in the range $(0.18 - 0.83)$. The high-z supernovae search team (HSST) discovered 14 supernovae in the redshift range $(0.16 - 0.62)$ and another 34 nearby supernovae [2] and used two separate methods for data fitting. Assuming $\Omega_m + \Omega_\Lambda = 1$, the analysis of this data gives

$$\Omega_m = 0.28 \pm 0.85(stat) \pm 0.05(syst).$$

The results obtained by both the supernova cosmological project and the

High -z Supernova Search Team present the strongest 'direct' evidence for a non-zero cosmological constant. However much work needs to be done both in understanding systematic uncertainties as well as Sn Ia properties before the case for a positive Λ is firmly established. One of the major causes of worry faced by 'standard candles' is source evolution and/or extinction. Clearly if the observed faintness of the supernova is being caused by physical processes rather than the accelerated expansion of the universe then all bets on $\Lambda > 0$ are off. Fortunately the effects of both evolution and extinction are likely to monotonically increase with redshift whereas the effect of Λ peaks at redshifts $z \sim 1$ and declines thereafter. Consequently observations conducted at redshifts just beyond $z = 1$ combined with lower disentangle source evolution/ extinction from effects due solely to Λ [33]

3.5.2 Gravitational Lensing and the luminosity distance

Consider an object of absolute luminosity $\mathcal{L}$ located at a co-ordinate distance r from an observer at $r = 0$. Light emitted by the object at a time t is received by the observer at $t = t_0$; t and t_0 being related by the cosmological redshift $1 + z = \dfrac{a(t_0)}{a(t)}$. The luminosity flux reaching the observer is

$$\mathcal{F} = \frac{\mathcal{L}}{4\pi \mathrm{d}_\mathrm{L}^2}\,, \tag{3.5.5}$$

where d_L is the luminosity distance to the object

$$d_L = a(t_0) r (1 + z)\,. \tag{3.5.6}$$

The luminosity distance d_L depends sensitively upon both the spatial curvature and the expansion dynamics of the universe. To demonstrate this we determine d_L using the expression for the co-ordinate distance r obtained by setting $ds^2 = 0$ in equation (1.2.1), resulting in

$$\int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = \int_t^{t_0} \frac{dt}{a(t)} = \eta_0 - \eta\,, \tag{3.5.7}$$

which gives

$$\begin{aligned} r =& \sin(\eta_0 - \eta)\,, && (k=+1) \\ =&(\eta_0 - \eta)\,, && (k=0) \\ =& \sinh(\eta_0 - \eta)\,, && (k=-1) \end{aligned} \tag{3.5.8}$$

where $\eta = \int_0^t cdt/a(t)$.
Furthermore, since $\dfrac{dz}{dt} = -(1+z)H(z)$, we get

$$\eta_0 - \eta = \int_t^{t_0} \frac{cdt}{a(t)} = \frac{c}{a_0 H_0} \int_0^z \frac{dz'}{h(z')}\,, \tag{3.5.9}$$

where $h(z) = H(z)/H_0$ and in a universe with several components

$$\frac{k}{a_0^2 H_0^2} = \Omega_{total} - 1\,. \tag{3.5.10}$$

Substituting equations(3.5.10) and (3.5.9) in equation (3.5.6) we get the following expression for the luminosity distance in a multicomponent universe with a cosmological term [34]

$$d_L(z) = \frac{(1+z)cH_0^{-1}}{|\Omega_{total} - 1|^{\frac{1}{2}}} S(\eta_0 - \eta)\,, \tag{3.5.11}$$

where

$$\eta_0 - \eta = |\Omega_{total} - 1|^{\frac{1}{2}} \int_0^z \frac{dz'}{h(z')}\,, \tag{3.5.12}$$

and $S(x)$ is defined as follows

$$S(x) = \sin(x) \qquad \text{if } k = 1\,(\Omega_{total} > 1),$$

$$S(x) = \sinh(x) \qquad \text{if } k = -1\,(\Omega_{total} < 1),$$

$$S(x) = x \qquad \text{if}\, k = 0 \;\; (\Omega_{total} = 1).$$

Before we turn to applications, let us consider a simple example which provides us with an insight into the role played by the luminosity distance d_L in cosmology. In a spatially flat universe the expression for d_L simplifies considerably, so that we get for the matter dominated model ($a \propto t^{\frac{2}{3}}$)

$$d_L^{MD} = 2cH_0^{-1}\left\{(1+z) - (1+z)^{\frac{1}{2}}\right\} \;. \tag{3.5.13}$$

On the other hand in de Sitter space ($a \propto exp(H_0 t)$)

$$d_L^{DS} = cH_0^{-1} z(1+z). \tag{3.5.14}$$

Comparing equations (3.5.13) and (3.5.14) we find $d_L^{DS}(z) > d_L^{MD}(z)$, which means that an object located at a fixed redshift will appear brighter in an Einstein-de Sitter universe than it will in de Sitter space (equivalently in the steady state model). In a spatially flat universe the presence of a Λ-term increases the luminosity distance to a given redshift, leading to interesting astrophysical consequences. Since the physical volume associated with a unit redshift interval increases in models with $\Lambda > 0$, the likelihood that light from a quasar will encounter a lensing galaxy is larger in such models. Consequently the probability that a quasar is lensed by intervening galaxies increases appreciably in a Λ dominated universe, and can be used as a test to constrain the value of Ω_Λ [35, 36]. The probability of a quasar at redshift

z_s being lensed relative to the fiducial Einstein-de Sitter model ($\Omega_m = 1$) is given by [34]

$$P(lens) = \frac{15}{4}\left[1 - \frac{1}{1+z_s}\right]^{-3} \int_0^{z_s} \frac{(1+z)^2 dz}{h(z)} \left[\frac{d(0,z)d(z,z_s)}{d(0,z_s)}\right]^2 , \quad (3.5.15)$$

where $d(z_1, z_2)$ is a generalization of the angular distance $d_A = d_L(1+z)^{-2}$

$$d(z_1, z_2) = \frac{1}{(1+z_2)|\Omega_{total} - 1|^{\frac{1}{2}}} S(\eta_{12}) , \quad (3.5.16)$$

where

$$\eta_{12} = \eta_1 - \eta_2 = |\Omega_{total} - 1|^{\frac{1}{2}} \int_{z_1}^{z_2} \frac{dz}{h(z)} , \quad (3.5.17)$$

and $S(\eta_{12})$ is defined as follows

$$\begin{aligned} S(\eta_{12}) &= \sin(\eta_{12}) && \text{if } k = 1\,(\Omega_{total} > 1), \\ S(\eta_{12}) &= \sinh(\eta_{12}) && \text{if } k = -1\,(\Omega_{total} < 1), \\ S(\eta_{12}) &= \eta_{12} && \text{if} k = 0\ \ (\Omega_{total} = 1). \end{aligned}$$

More formally, one can compute the probability for a source at redshift z_s being lensed in a $\Omega_\Lambda + \Omega_m = 1$ universe (relative to the corresponding probability in a $\Omega_m = 1, \Omega_\Lambda = 0$ model). This relative probability is nearly five times large at $z_s = 1$ and about thirteen times larger for $z_s = 2$ in a purely cosmological constant dominated universe [34, 36, 37]. This effect quantifies the fact that the physical volume associated with unit redshift interval is larger in models with cosmological constant and hence the probability that a light ray will encounter a galaxy is larger in these cases.

Current analysis of lensing data gives somewhat differing constraints based on the assumptions which are made [38]; but typically all these constraints are about $\Omega_\Lambda \lesssim 0.7$. The result [39] from Cosmic Lens All Sky Survey (CLASS), for example, it gives $\Omega m = 0.31^{+0.27}_{-0.14}(68\%)^{+0.12}_{-0.10}(\text{systematic})$ for a $k = 0$ universe.

3.5.3 Age of the universe and cosmological constant

From equation (3.4.1) we can also determine the current age of the universe by the integral

$$H_0 t_0 = \int_0^1 \frac{dq}{\sqrt{2(E-V)}} \,. \tag{3.5.18}$$

Since most of the contribution to this integral comes from late times, we can ignore the radiation term and set $\Omega_r \approx 0$. When both Ω_m and Ω_Λ are present and are arbitrary, the age of the universe is determined by the integral

$$\begin{aligned} H_0 t_0 &= \int_0^\infty \frac{dz}{(1+z)\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}} \\ &\approx \frac{2}{3}(0.7\Omega_m - 0.3\Omega_\Lambda + 0.3)^{-0.3} \,. \end{aligned} \tag{3.5.19}$$

The integral, which cannot be expressed in terms of elementary functions, is well approximated by the numerical fit given in the second line. It is obvious that, for a given Ω_m, the age is higher for models with $\Omega_\Lambda \neq 0$.

If the universe is populated by dust-like matter (with $w = 0$) and another component with an equation of state parameter w_x, then the age of the universe will again be given an integral similar to the one in equation (3.5.19) with Ω_Λ replaced by $\Omega_x(1+z)^{3(1+w_x)}$. This will give

$$\begin{aligned} H_0 t_0 &= \int_0^\infty \frac{dz}{(1+z)\sqrt{\Omega_m(1+z^3)+\Omega_x(1+z)^{3(1+w_x)}}} \\ &= \int_0^1 dq \left(\frac{q}{\Omega_m + \Omega_x q^{-3w_x}}\right)^{\frac{1}{2}} . \end{aligned} \tag{3.5.20}$$

The integrand varies from 0 to $(\Omega_m+\Omega_x)^{-\frac{1}{2}}$ in the range of integration for $w<0$ with the rapidity of variation decided by w. As a result, $H_0 t_0$ increases rapidly as w changes from 0 to -3 or so and then saturates to a plateau. Even an absurdly negative value for w like $w=-100$ gives $H_0 t_0$ of only about 1.48. This shows that even if some exotic dark energy is present in the universe with a constant, negative w, it cannot increase the age of the universe beyond about $H_0 t_0 \approx 1.48$.

The comments made above pertain to the current age of the universe. It is also possible to obtain an expression similar to equation (3.5.18) for the age of the universe at any given redshift z

$$H_0 t(z) = \int_z^\infty \frac{dz'}{(1+z')h(z')}\,; \qquad h(z) = \frac{H(z)}{H_0}\,, \tag{3.5.21}$$

and use it to constrain Ω_Λ. For example, the existence of high redshift galaxies with evolved stellar population, high redshift radio galaxies and age dating of high redshift QSOs can all be used in conjunction with this equation to put constrains on Ω_Λ. Most of these observations require either $\Omega_\Lambda \neq 0$ or $\Omega_{tot} < 1$ if they have to be consistent with $h \gtrsim 0.6$. Unfortunately, the interpretation of these observations at present requires fairly complex modeling and hence the results are not water tight.

3.6 Models with evolving cosmological "constant"

The observations which suggest the existence of non-zero cosmological constant raises serious theoretical problems. These difficulties have led people to consider the possibility that the dark energy in the universe is not just a cosmological constant but is of more complicated nature, evolving with time. Its value today can then be more naturally set by the current expansion rate rather then predetermined earlier on- thereby providing a solution to the cosmological constant problems.

Through a hot of models have been constructed based on this hope, none of them provides a satisfactory solution to the problems of fine-tuning. Moreover, all of them involve an evolving equation of state parameter $w_x(a)$ for the unknown ("x") dark energy component, there by taking away all predictive power from cosmology [13]. Ultimately, however, this issue needs to settled observationally by checking whether $w_x(a)$ is a constant [equal to -1, for the cosmological constant] at all epochs or whether it is indeed varying with a. We shall now discuss several observational and theoretical issues connected with this theme. The complete knowledge of the T^a_b uniquely determines $H(a)$, but the converse is not true. If we know only the function $H(a)$, it is not possible to determine the nature of the energy density which is present in the universe. We have already seen that geometrical measurements can only provide, at best, the functional form of $H(a)$. It follows that purely geometrical measurements of the Friedmann universe will never allow us to determine the material content of the universe.

The only exception to this rule is when we assume that each of the components in the universe has constant w_i. This is fairly strong assumption and in fact, will allow us to determine the components of the universe from the knowledge of the function $H(a)$. To see this, we first note that the term (k/a^2) in equation (1.2.14) can be thought of as contributed by a hypothetical species of matter with $w = -\frac{1}{3}$. Hence equation (1.2.14) can be written in the form

$$\frac{\dot{a}^2}{a^2} = H_0^2 \sum_i \Omega_i (\frac{a_0}{a})^{3(1+w_i)} \,, \tag{3.6.1}$$

with a term having $w_i = -\frac{1}{3}$ added to the sum. Let $\alpha \equiv 3(1+w)$ and $\Omega(\alpha)$ denote the fraction of the critical density contributed by matter with $w = (\frac{\alpha}{3}) - 1$. (For discrete values of w_i and α_i, the function $\Omega(\alpha)$ will be a sum of Dirac delta functions). In the continuum limit, equation (3.6.1) can be rewritten as

$$H^2 = H_0^2 \int_{-\infty}^{\infty} d\alpha \; \Omega(\alpha) e^{-\alpha q} \,, \tag{3.6.2}$$

where $(\frac{a}{a_0}) = exp(q)$. The function $\Omega(\alpha)$ is assumed to have finite support for the expression on the right hand side to converge. If the observations determine the function $H(a)$, then the left hand side can be expressed as a function of q. An inverse Laplace transform of this equation will then determine the form of $\Omega(\alpha)$ there by determining the composition of the universe, as long as all matter can be described by an equation of state of the form $p_i = w_i \rho_i$ with w_i =constant for all $i = 1, 2, \ldots N$.

More realistically one is interested in models which has a complicated form of $w_x(a)$ for which the above analysis is not applicable. Let us divide the source energy density into two components: $\rho_k(a)$, which is known from independent observations and a component $\rho_x(a)$ which is not known. From equation (1.2.14), it follows that

$$\frac{8\pi G}{3}\rho_x(a) = H^2(a)\{1 - Q(a)\}; \quad Q(a) \equiv \frac{8\pi G\rho_k(a)}{3H^2(a)} . \tag{3.6.3}$$

Now, $T^a_b;_a = 0$ translates to the condition $d(\rho_i a^3) = -p_i da^3$. It follows that the evolution of the energy densities of each component is essentially dependent on the parameter $w_i \equiv (\frac{p_i}{\rho_i})$ which, in general, could be a function of time. Integrating $d(\rho_i a^3) = -w_i\rho_i da^3$, we get

$$\rho_i = \rho_i(a_0)(\frac{a_0}{a})^3 exp\left[-3\int_{a_0}^{a}\frac{d\bar{a}}{\bar{a}}w_i(\bar{a})\right] . \tag{3.6.4}$$

Taking derivative of $\ln\rho_x(a)$ and using equation (3.6.4), it is easy to obtain the relation

$$w_x(a) = -\frac{1}{3}\frac{d}{d\ln a}\ln\left[\{1 - Q(a)\}H^2(a)a^3\right] \tag{3.6.5}$$

In geometrical observations of the universe give us $H(a)$ and other observations give us $\rho_k(a)$ then one can determine Q and thus $w_x(a)$. While this is possible, in principle the uncertainties in measuring both H and Q makes this a nearly impossible route to follow in practice. In particular, one is interested in knowing whether w evolves with time or a constant and this turns out to be a very difficult task observationally.

3.7 Cosmological constant Λ from String theory

A relativistic point particle is a zero dimensional object; the world line of such a particle, describing its time evolution, will be one dimensional and the standard quantum field theory (like QED) uses real and virtual world lines of particles in its description. In contrast, a string (at a given moment of time) will be described by an one dimensional entity and its time evolution will be a two dimensional world surface called the world sheet. The basic formalism of string theory- considered to be a possible candidate for a model for quantum gravity- uses a two dimensional world sheet rather than the one dimensional world line of a particle to describe fundamental physics. Since the point particle has been replaced by a more extended structure, string theory can be manifest as low energy particles. This provides a hope for describing both gauge theories and gravity in a unified manner.

According to the No-Go theorem [40] it is not possible to find de-Sitter solutions only in the presence of the lowest order terms in the 10 and 11 dimensional super gravity action. However this situation is improved when α' or quantum corrections to the three-level action are taken into account or extended objects like D-branes are present. In fact Kachru, Kallosh, Linde and Trivedi (KKLT) [41] constructed de-Sitter vacua by incorporating non-perturbative corrections to a superpotential in the context of type IIB string theory compactified on a Cabbi-Yau manifold in the presence of flux. The importance of flux to insolving the cosmological constant problem was originally realized in ref. [42]. In what follows we shall briefly discuss the effect of a four-form gauge flux to construct de-Sitter vacua [42] and then proceed to the review of the KKLT scenario using flux compactification.

3.7.1 Four-form fluxes and quantization

Let us consider a four-form flux field $F_4^2 = F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}$ which appears in M-theory. The starting point is a four dimensional gravity action in the presence of a negative bare cosmological constant Λ_b and the four form flux field [42]

$$S = \int d^4x\sqrt{-g}(\frac{1}{2k^2}R + \Lambda_b - \frac{1}{2.\,4!}F_4^2), \tag{3.7.1}$$

which arises as an effective action arising, e.g., from a $M^4 \times S^7$ compactification. The bare cosmological constant is should be negative at the perturbative regime of string theory if it exists.

The four-form equation of motion, $\nabla_\mu(\sqrt{-g}F^{\mu\nu\rho\sigma}) = 0$, gives the solution $F^{\mu\nu\rho\sigma} = c\epsilon^{\mu\nu\rho\sigma}$, where $F^{\mu\nu\rho\sigma}$ is an antisymmetric tensor with c being constant. Since $F_4^2 = -24c^2$, we find that the effective cosmological constant is given by

$$\Lambda = -\Lambda_b - \frac{1}{48}F_4^2 = -\Lambda_b + \frac{c^2}{2}\,. \tag{3.7.2}$$

This shows that it is possible to explain a small value of Λ provided that the bare cosmological constant is nearly canceled by the term coming from the four-form flux. However as long as the contribution of the flux is continuous, one can not naturally obtain the observed value of Λ.

Bousso and Polchinski tackled this problem by quantizing the value of c [42]. This implies that c is discontinuous as $c = nq$, where n is an integer.

Although a single flux is not sufficient to explain the small values of Λ because of the large steps involved, this situation is improved by considering j multiple fluxes. In this case an effective cosmological constant is given by

$$\Lambda = -\Lambda_b + \frac{1}{2}\sum_{i=1}^{j} n_i^2 q_i^2 \,. \tag{3.7.3}$$

It was shown that one can explain the observed value of Λ with j of order 100, which is not unrealistic [42]. The work of Bousso and Polchinski did not address the problem of the stabilization of the modulus fields, but it opened up a new possibility for constructing large numbers of de-Sitter vacua using fluxes- and this has been called the "string landscape".

3.7.2 The KKLT scenario

KKLT [41] provided a mechanism to construct de-Sitter vacua of type IIB string theory based on flux compactifications on a Calabi- You manifold. They first of all fixed all the moduli associated with the compactification in an anti de-Sitter vacua by preserving super symmetry. Then they incorporated nonperturbative corrections to the superpotential to obtain de-Sitter vacua.

The low energy effective action of string/M-theory in four dimensions is described by $N = 1$ supergravity

$$\begin{aligned} S = \int d^4x\sqrt{-g}[\frac{M_{pl}^2}{2}R + g^{\mu\nu}K_{\alpha\bar{\beta}}\partial_\mu\psi^\alpha\partial_\nu\bar{\psi}^\beta - \\ e^{K/M_{pl}^2}(K^{\alpha\bar{\beta}}D_\alpha W D_{\bar{\beta}}\bar{W} - \frac{3}{M_{pl}^2}|W|^2)], \end{aligned} \tag{3.7.4}$$

where α, β run over all moduli fields ψ. Here $W(\psi^\alpha)$ and $K(\psi^\alpha, \bar{\psi}^\beta)$ are the superpotential and the Kahler potential, respectively, and

$$K_{\alpha\bar{\beta}} = \frac{\partial^2 K}{\partial\psi^\alpha \partial\bar{\psi}^\beta}\,, \qquad D_\alpha W \equiv \frac{\partial W}{\partial\psi^\alpha} + \frac{W}{M_{pl}^2}\frac{\partial K}{\partial\psi^\alpha}. \tag{3.7.5}$$

The supersymmetry is unbroken only for the vacua in which $D_\alpha W = 0$ for all α, which means that the effective cosmological constant is not positive from the action given by equation (3.7.4).

S.Giddings, S.Kachru and J.Polchinski adopted the following tree level functions for K and W in the flux compactification of type IIB string theory [43].

$$K = -3\ln[-i(\rho-\bar{\rho})] - \ln[-i(\tau-\bar{\tau})] - \ln[-i\int_\mu \Omega\Lambda\bar{\Omega}], \tag{3.7.6}$$

$$W = \int_\mu G_3\Lambda\Omega, \tag{3.7.7}$$

where ρ is the volume modulus which includes the volume of the Calabi-Yau space and an axion coming from the R-R four-form $C_{(4)}$, and $\tau = C_{(0)} + ie^{-\phi}$ is the axion-dilation modulus. σ is the holomorphic three-form on the Calabi-Yau space G_3 is defined by $G_3 = F_3 - \tau H_3$ where F_3 and H_3 are the R-R flux and the NS-NS flux, respectively, on the 3-cycles of the internal Calabi-Yau manifold μ. Since W is not a function of ρ, we obtain $K^{\rho\bar{\rho}} D_\rho W D_{\bar{\rho}}\bar{W} = 3|W|^2$, then equation (3.7.4) gives the supergravity potential

$$V = e^K (K^{i\bar{j}} D_i W D_{\bar{j}}\bar{W}), \tag{3.7.8}$$

where i, j run over all moduli fields except for ρ. The condition $D_i W = 0$ fixes all complex moduli except for ρ [43], which gives a zero effective cosmological constant. On the other hand, the supersymmetric vacua satisfying $D_\rho W = 0$ gives $W = 0$, where as, the non supersymmetric vacua yield $W = W_0 \neq 0$.

To fix the volume modulus ρ as well, KKLT [41] added a non-perturbative correction [44] to the superpotential, which is given by

$$W = W_0 + Ae^{ia\rho}, \tag{3.7.9}$$

where A and a are constants and $W_0 \equiv \int G_3 \Lambda \Omega$ is the tree level contribution. This correction is actually related to the effect of brane instantons. Note that the conditions $D_i W = 0$ are automatically satisfied. For simplicity we set the the axion-dilaton modulus to be zero and take $\rho = i\sigma$. Taking real values of A, a and W_0, we find that the supersymmetric condition $D_\rho W = 0$ gives

$$W_0 = -Ae^{-a\sigma_c}(1 + \frac{2}{3}a\sigma_c), \tag{3.7.10}$$

which fixes the volume modulus ρ in terms of W_0. This produces the anti De-Sitter vacua, that is

$$V_{Ads} = -3e^K |W|^2 = -\frac{a^2 A^2 e^{-2a\sigma_c}}{6\sigma_c}\,. \tag{3.7.11}$$

Hence all the moduli are stabilized while preserving supersymmetry with a negative cosmological constant.

In order to obtain a De-Sitter vacuum, KKLT introduced an anti-D3 brane in a warped background. Since fluxes F_3 and H_3 are also the sources for a warp factor [43], models with fluxes generically correspond to a warped compactification, whose metric is given by

$$ds_{10}^2 = e^{2B(y)} g_{\mu\nu}(x) dx^\mu dx^\nu + e^{-2B(y)} \tilde{g}_{mn}(y) dy^m dy^n\,, \tag{3.7.12}$$

where the factor, e^B can be computed in the regions closed to a conifold singularity of the Calabi-You manifold. This warp factor is exponentially suppressed at the tip of the throat, depending on the fluxes as

$$E^{B_{min}} \sim \exp\left(-\frac{2\pi N}{3g_s M}\right), \tag{3.7.13}$$

where g_s is the string coupling, integers M and N are the R-R and NS-NS three-form flux, respectively. While the warp factor is of order one at generic points in the y-space, its minimum value can be extremely small for a suitable choice of fluxes.

The background fluxes generate a potential for the world-volume scalars of the anti-D3 brane, which means that they do not introduce additional moduli [45]. The anti-D3 brane, however, provides an additional energy to the supergravity potential [41, 45],

$$\delta V = \frac{2b_0^4 T_3}{g_s^4}\frac{1}{(Im\rho)^3}, \tag{3.7.14}$$

where T_3 is the brane tension and b_0 is the warp factor at the location of the anti-D3 brane. The anti-D3 brane energetically prefers to sit at the tip of the throat, giving $b_0 = e^{B_{min}}$. The total potential is the sum of equations (3.7.11)and (3.7.14), that is

$$V = \frac{2b_0^4 T_3}{g_s^4}\frac{1}{(Im\rho)^3} - \frac{a^2 A^2 e^{-2a\sigma_c}}{6\sigma_c}. \tag{3.7.15}$$

Then one can obtain positive cosmological constant by tuning the flux integers M and N.

The life time of the vacua was found to be larger than the age of the universe and hence these solutions can be considered as stable for practical purposes [41]. Although a fine tuning problem of Λ still remains in this scenario, it is interesting that string theory in principle gives rise to a stable De-Sitter vacua with all moduli fixed.

3.7.3 Relaxation of Λ in string theory

J.L.Feng, J.March-Russel, S.Sethi and F.Wilczek [46] developed the earlier approached by J.D.Brown and C.Teitelboim [47] to relax the effective cosmological constant through the nucleation of branes coupled to a three-index gauge potential. The influence of string theory in the new approach is important, the brane depends on the compactification of the extra dimensions which in turn can provide the required very small quantized unit for jumps in the effective cosmological term. As well as this feature, when considering multiple coincident branes [46], the author show that the internal degrees of freedom for such a configuration can dramatically large density of states factors.

For consistency, the dynamics of the system must be such that the cosmological constant relaxes quickly enough from high energy scales; but today remains stable on a time scale of the universe, a constraint which leads to a non-trivial relation between the scale of supersymmetry breaking and the value of the cosmological constant. In particular the constraint becomes

$$M_{SUSY}^2 \leqslant (10^{-3}eV)(M_{planck}), \tag{3.7.16}$$

which rules out large supersymmetry breaking scales for these relaxation models, with the largest possible scale still viable in nature. Time will tell whether the relaxation mechanism is sufficiently versatile to uniquely pick out the actual vacuum we live in, but it is certainly a novel approach to determining it.

3.8 Reinterpreting the cosmological constant

It is possible to study the cosmological constant from various other directions in which the mathematical structure of equation (3.3.3) is reinterpreted differently. Though none of these ideas have been developed into a successful formal theory, they might contain ingredients which may eventually provide a solution to this problem.

3.8.1 Cosmological constant as Lagrange multiplier

The action principle for gravity in presence of a cosmological constant

$$\begin{aligned} A =& \frac{1}{16\pi G}\int (R-2\Lambda)\sqrt{-g}\; d^4x \\ =& \frac{1}{16\pi G}\int R\sqrt{-g}\; d^4x - \frac{\Lambda}{8\pi G})\int \sqrt{-g}\; d^4x\,. \end{aligned} \tag{3.8.1}$$

can be thought of as a variational principle extremizing the integral over R, subject to the condition that the four-volume of the universe remains constant. To implement the constraint that the four-volume is a constant, one will add a Lagrange multiplier term which is identical in structure to the second term in the above equation. Hence, mathematically one can think of the cosmological constant as a Lagrange multiplier ensuring the constancy of the four-volume of the universe when the metric is varied. If we take this interpretation, then it is necessary to specify four-volume of the universe before the variation is performed and determine the cosmological constant so that the four-volume has this specified volume. A Friedmann model with positive cosmological constant in Minkowski space will lead to a finite 3-volume proportional to $\Lambda^{-3/2}$ on spatial integration. The time integration, however, has an arbitrary range and one needs to restrict the integration to part of this range by invoking some physical principle. If we take this to be typically the age of the universe, then we will obtain a time dependent cosmological constant $\Lambda(t)$ with $\Lambda(t)H(t^{-2})$ remaining of order unity. While this appears to be a conceptually attractive idea, it is difficult to impliment it in a theoretical model.

3.8.2 Cosmological constant as a constant of integration

Cosmological constant as a constant of integration, this idea does not solve the problem in the sense that it still leaves its value undetermined. But this changes the perspective and allows one to think of the cosmological constant as a non dynamical entity. [48]

One simple way of achieving this is to assume that the determinant g of g_{ab} is not dynamical and admit only those variations which obeys the condition $g^{ab}\delta g_{ab} = 0$ in the action principle. This is equivalent to eliminating the trace part of Einstein's equations. Instead of the standard result, we will now be led to the equation

$$R^i_k - \frac{1}{4}\delta^i_k R = 8\pi G(T^i_k - \frac{1}{4}\delta^i_k T)\,, \tag{3.8.2}$$

which is just the traceless part of Einstein's equation. The general covariance of the action, however implies that $T^{ab}_{;b} = 0$ and the Bianchi identities $(R^i_k - \frac{1}{2}\delta^i_k R);_i = 0$ continue to hold. These two conditions imply that $\partial_i R = -8\pi G \partial_i T$ requiring $r + 8\pi GT$ to be a constant. Calling this constant

(-4Λ) and combining with equation (3.8.2), we get

$$R^i_k - \frac{1}{2}\delta^i_k R - \delta^i_k \Lambda = 8\pi G T^i_k \,, \tag{3.8.3}$$

which is precisely Einstein's equation in the presence of cosmological constant. In this approach, the cosmological constant has nothing to do with any term in the action or vacuum fluctuations and is merely an integration constant. Like any other integration constant its value can be fixed by invoking suitable boundary conditions for the solutions.

Chapter 4

Scalar field models of Dark Energy

Introduction

The cosmological constant corresponds to a fluid with a constant equation of state $w_{\Lambda} = \frac{p_{\Lambda}}{\rho_{\Lambda}} = -1$ as $\rho_{\Lambda} = \frac{\Lambda}{8\pi G}$ and $p_{\Lambda} = -\frac{\Lambda}{8\pi G}$. Now, the observations which constrain the value of w_{Λ} today to be close to that of the cosmological constant, these observations actually say relatively little about the time evolution of w_{Λ}, and so we can broaden our horizons and consider a situation in which the equation of state of dark energy changes with time, such as in inflationary cosmology. Scalar fields naturally arise in particle physics including string theory and these can act as candidates for dark energy. So far a wide variety of scalar field dark energy models have been proposed. These include Quintessence, K-essence, Tachyon and phantom dark energy amongst many.

4.1 Cosmological model and scalar field equations

We consider the homogeneous and isotropic universe with metrics of 4-space

$$ds^2 = g_{ij}dx^i dx^j = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \quad (4.1.1)$$

where $k = +1, 0, -1$ for Riemannian, Euclidean and Lobachevskian 3-space respectively. The factor $a(t)$ is a radius of 3-sphere ($k = +1$), 3-pseudosphere ($k = -1$) or the scale factor, normalized to 1 at current epoch ($k = 0$). For $c = 1$, the time variable $t = x_0$ has the dimension of length. If the universe is filled with dust matter and dark energy, the dynamics of its expansion is completely described by the Einstein equations (1.2.9) as

$$R_{ij} - \frac{1}{2}g_{ij}R = 8\pi G(T_{ij}^{(m)} + T_{ij}^{(de)}), \quad (4.1.2)$$

where R_{ij} is the Ricci tensor and $T_{ij}^{(m)}, T_{ij}^{(de)}$ are energy-momentum tensor of matter (m) and dark energy (de). If these components interact only gravitationaly then each of them satisfy energy-momentum conservation law separately

$$T_{j;i}^{i(m,de)} = 0\,. \quad (4.1.3)$$

For ideal fluid with density $\rho_{(m,de)}$ and pressure $p_{(m,de)}$, related by the equation of state $p_{(m,de)} = w_{(m,de)}\rho_{(m,de)}$, it gives

$$\dot{\rho}_{m,de} = -3\frac{\dot{a}}{a}\rho_{_{(m,de)}}(1 + w_{(m,de)})\,. \quad (4.1.4)$$

For constant $w_{(m,de)}$ these equations are easily integrated and dependences of dust matter and dark energy densities on the scale factor are simply obtained

$$\rho_m = \rho_m^{(o)}(\frac{a}{a_o})^{-3}, \qquad \rho_{de} = \rho_{de}^{(o)}(\frac{a}{a_o})^{-3(1+w_{de})}. \tag{4.1.5}$$

here "o" denotes the present values. The matter is considered to be non-relativistic, so $w_m = 0$.

Let the dark energy be a scalar field $\phi(X,t)$ with classical Lagrangian

$$L = \frac{1}{2}\phi_{;i}\phi^{;i} - V(\phi), \tag{4.1.6}$$

where $V(\phi)$ potential energy density or the field potential. We suppose also the scalar field to be homogeneous in expanding homogeneous isotropic universe $[\phi(X,t) = \phi(t)]$, so its energy density and presure depends only on time

$$\rho_{de}(t) = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad p_{de}(t) = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{4.1.7}$$

Then the conservation law given by equation (1.3.3) gives the scalar field evolution equation

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi},$$

where $H = \frac{\dot{a}}{a}$ is the Hubble constant for any moment of time t.

If the Lagrangian of scalar field has the form

$$L = -V(\phi)\sqrt{1 - \phi_{;i}\phi^{;i}}, \tag{4.1.8}$$

such field is called the tachyon one. The energy density and pressure of the homogeneous tachyon field are defined as follows

$$\rho_{de} = \frac{V(\phi)}{\sqrt{1-\dot{\phi}^2}}, \qquad p_{de} = -V(\phi)\sqrt{1-\dot{\phi}^2}. \tag{4.1.9}$$

The conservation equation (1.3.3) describes the evolution of the tachyon field in a such way

$$\ddot{\phi} + (1 - \dot{\phi}^2)(\frac{1}{V}\frac{dV}{d\phi} + 3H\dot{\phi}) = 0. \qquad (4.1.10)$$

Now we define the dimensionless densities Ω_m, Ω_{de} and Ω_k

$$\Omega_m = \frac{\rho_m^{(o)}}{\rho_{cr}^{(o)}}, \quad \Omega_{de} = \frac{\rho_{de}^{(o)}}{\rho_{cr}^{(o)}}, \quad \Omega_k = -\frac{ka_o^{-2}}{H_o^2},$$

where $\rho_{cr}^{(o)} = \dfrac{3H_o^2}{8\pi G}$ is critical density of the universe at the present epoch and ka_o^{-2}, the 3-space curvature. Using them, the Einstein equations for the model of the universe with dust matter, dark energy and curvature can be written in the form

$$H = H_o\sqrt{\Omega_m(\frac{a_o}{a})^3 + \Omega_k(\frac{a_o}{a})^2 + \Omega_{de}(\frac{a_o}{a})^{3(1+w_{de})}}, \qquad (4.1.11)$$

$$q = \frac{1}{2}\frac{\Omega_m(\frac{a_o}{a})^3 + (1+3w_{de})\Omega_{de}(\frac{a_o}{a})^{3(1+w_{de})}}{\Omega_m(\frac{a_o}{a})^3 + \Omega_k(\frac{a_o}{a})^2 + \Omega_{de}(\frac{a_o}{a})^{3(1+w_{de})}}, \qquad (4.1.12)$$

where $q = -\dfrac{\ddot{a}}{(aH^2)}$, the acceleration parameter for any time t.

The first equation rewritten for the present time to gives the equality $\Omega_{de} + \Omega_m + \Omega_k = 1$. From the second one it follows that the accelerated expansion $(q < 0)$ is possible only when $w_{de} < -\dfrac{1}{3}$. At the early stages of the evolution of the universe $q > 0$ always and $q \longrightarrow 0.5$ when $a \longrightarrow 0$. The change of acceleration sign (moment when $q = 0$) or transition from deceleration to acceleration occurs at redshift Z_q which depends on the relation between the matter and dark energy densities $Z_q = \left[-(1+3w_{de})\dfrac{\Omega_{de}}{\Omega_m}\right]^{\frac{1}{3w_{de}}} - 1$. The scalar field density begins to dominate $(\rho_{de} \geq \rho_m)$, when $Z_{de} = (\dfrac{\Omega_{de}}{\Omega_m})^{\frac{1}{(3w_{de})}} - 1$.

To be a candidate for dark energy the scalar field with Lagrangian given by equation (4.1.6) must satisfy the condition $\dot{\phi}^2 \leq V(\phi)$. If $\dot{\phi}^2 \ll V(\phi)$ then we will have the scalar field analogous to cosmological constant in the Einstein equations $w_{de} = -1$. For field with Lagrangian given by equation (4.1.8), the expansion accelerates when $0 \leq \dot{\phi}^2 < \frac{2}{3}$. The scalar field with Lagrangian equation (4.1.6) which satisfies the conditions $\dot{\phi}^2 > 0$ and $-1 \leq w_{de} \leq -\frac{1}{3}$ is called Quintessence and cosmological models with $\Omega_{de} > \Omega_{cdm} > \Omega_b$ are noted by QCDM.

4.2 Quintessence

Quintessence is described by an ordinary scalar field ϕ minimally coupled to gravity, but as we will see with particular potentials that lead to late time inflation. The action for quintessence is given by

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}(\nabla\phi)^2 - V(\phi)\right], \tag{4.2.1}$$

where $V(\phi)$ is the potential of the field and $(\nabla\phi)^2 = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. The spatially flat and homogeneous cosmological model is given by

$$ds^2 = dt^2 - a^2(t)[dx^2 + dy^2 + dz^2]. \tag{4.2.2}$$

In this space-time $\phi(x,t) = \phi(t)$ due to homogeneity. Using equation (4.2.2), the variation of the action given by equation (4.2.1) with respect to ϕ gives

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0\,. \tag{4.2.3}$$

The energy momentum tensor of the field is derived by varying action equation (4.2.1) in terms of $g^{\mu\nu}$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\,. \tag{4.2.4}$$

Taking $\delta\sqrt{-g} = -(\frac{1}{2})\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$, we find

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right]. \tag{4.2.5}$$

In the flat Friedmann background we obtain the energy density and pressure density of the scalar field

$$\begin{aligned} \rho = -T_0^0 &= \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \\ p = T_i^i &= \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \end{aligned} \tag{4.2.6}$$

Then equations (1.2.14) and (1.2.17) yield

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right], \tag{4.2.7}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 - V(\phi)\right]. \tag{4.2.8}$$

From equation (4.2.8) we find that the universe accelerates for $\dot{\phi}^2 < V(\phi)$. This means that one requires a flat potential to give rise to an requires a flat potential to give rise to an accelerated expansion. In the context of inflation the slow-roll parameters

$$\epsilon = \frac{m_{pl}^2}{16\pi}\left(\frac{1}{V}\frac{dV}{d\phi}\right)^2, \qquad \eta = \frac{m_{pl}^2}{8\pi}\frac{1}{V}\frac{d^2V}{d\phi^2}\,, \tag{4.2.9}$$

are often used to check the existence of an inflationary solution for the model given by equation (4.2.1). Inflation occurs if the slow-roll conditions, $\epsilon \ll 1$ and $|\eta| \ll 1$, are satisfied.

The equation of state for the field ϕ is given by

$$w_\phi = \frac{p}{\rho} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)} \,. \tag{4.2.10}$$

Expressed in terms of w_ϕ, equations (4.2.6) may be put in the form

$$\frac{\dot{\phi}^2}{\rho} = 1 + w_\phi \,, \tag{4.2.11}$$

$$\frac{2V(\phi)}{\rho} = 1 - w_\phi \,. \tag{4.2.12}$$

Connecting conservation equations (1.3.3) and (4.2.10), it is obtained that

$$\dot{\rho} + 3H(1 + w_\phi)\rho = 0 \,, \tag{4.2.13}$$

where $H = \frac{\dot{a}}{a}$. Equation (4.2.13) integrates to

$$\rho = \rho^0 \left(\frac{a_0}{a}\right)^{3(1+w_\phi)} \,. \tag{4.2.14}$$

So, Friedmann equation (1.2.14) is obtained

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho^0 \left(\frac{a_0}{a}\right)^{3(1+w_\phi)} \,, \tag{4.2.15}$$

yielding the solution

$$a(t) = a_0 \left[1 + \frac{3(1 + w_\phi)}{2} H_0 \sqrt{0.73}(t - t_0)\right]^{\frac{2}{3(1+w_\phi)}} \,. \tag{4.2.16}$$

Using WMAP result $\rho^0 = 0.73\rho_c^0$, where ρ_c^0 is the critical density of the universe at the present epoch.

As $-1 < w_\phi < -\frac{1}{3}$, $\frac{2}{3(1+w_\phi)} > 1$. So equation (4.2.16) yields $\ddot{a} > 0$ showing accelerated expansion.

From the Raychoudhuri's equation (1.2.15), we obtain the relation

$$\dot{H} = -4\pi G(\rho + p) = -4\pi G\dot{\phi}^2 \,, \tag{4.2.17}$$

using equation (4.2.6).

Now $V(\phi)$ and $\dot{\phi}$ can be expressed in terms of H and $\dot{H}$ as

$$V = \frac{3H^2}{8\pi G}\left(1 + \frac{\dot{H}}{3H^2}\right), \tag{4.2.18}$$

and

$$\phi = \int dt \left(-\frac{\dot{H}}{4\pi G}\right)^{\frac{1}{2}} . \tag{4.2.19}$$

From equation (4.2.16)

$$H = -H_0\sqrt{0.73}\left[1 - \frac{3(1+w_\phi)}{2}H_0\sqrt{0.73}(t - t_0)\right]^{-1} . \tag{4.2.20}$$

Using H in quation (4.2.19) and integrating,

$$\phi = \sqrt{\frac{2}{3(1+w_\phi)}}\ln\left[1 - \frac{3(1+w_\phi)}{2}H_0\sqrt{0.73}(t - t_0)\right] . \tag{4.2.21}$$

Thus, connecting equations (4.2.18)-(4.2.21)

$$V(\phi) = \rho^0\left[\frac{1+w_\phi}{2}\right]\exp\left[-\sqrt{24\pi(1+w_\phi)G\phi}\right] . \tag{4.2.22}$$

It shows that exponential potential can be used for Quintessence scalar if $\frac{2}{3(1+w_\phi)} > 1$.

4.3 K-essence

Quintessence relies on the potential energy of scalar fields to lead to the late time acceleration of the universe. It is possible to have a situation where the accelerated expansion arises out of modifications to the kinetic energy of the scalar fields. In 1999, Armendariz-picon, T. Damour and V. Mukhanov proposed kinetically driven inflation, [11] called K-inflation to explain early universe inflation at high energies. This scenario was first applied to dark energy by Chiba et al. [12]. The analysis was extended to a more general Lagrangian by Armendariz-Picon et al. [49] and this scenario was called "K-essence"

K-essence is characterized by a scalar field with a non-canonical kinetic energy. The most general scalar-field action which is a function of $\tilde{\phi}$ and $\tilde{X} \equiv -(\frac{1}{2})(\bigtriangledown\tilde{\phi})^2$ is given by

$$S = \int d^4x\sqrt{-g}\tilde{\mathcal{L}}(\tilde{\phi}, \tilde{X})\,, \tag{4.3.1}$$

where the Lagrangian density $\tilde{\mathcal{L}}(\tilde{\phi}, \tilde{X})$ corresponds to a pressure density. Usually K-essence models are restricted to the Lagrangian density of the form [12, 49]

$$\tilde{\mathcal{L}}(\tilde{\phi}, \tilde{X}) = f(\tilde{\phi})L(\tilde{X})\,. \tag{4.3.2}$$

One of the motivations to consider this type of Lagrangian originates from string theory [11]. The four-dimensional effective string action is generally given by

$$S = \int d^4x\sqrt{-\tilde{g}}\{B_g(\tilde{\phi})\tilde{R} + B^{(0)}_{\tilde{\phi}}(\tilde{\phi})(\tilde{\nabla}\tilde{\phi})^2 - \alpha'\left[C^{(1)}_1 B^{(1)}_{\tilde{\phi}}(\tilde{\phi})(\tilde{\nabla}\tilde{\phi})^4 + \ldots\right] + \mathcal{O}(\alpha'^2)\}, \tag{4.3.3}$$

where $\tilde{\phi}$ is the dilaton field that controls the strength of the string coupling $g_s^2 = e^{\tilde{\phi}}$ [50]. Here we set $k^2 = 8\pi G = 1$. In the weak coupling regime ($e^{\tilde{\phi}} \ll 1$) the coupling functions have the dependence $B_g \simeq B^{(o)}_{\tilde{\phi}} \simeq B^{(1)}_{\tilde{\phi}} \simeq e^{-\tilde{\phi}}$. If we make a conformal transformation $g_{\mu\nu} = B_g(\tilde{\phi})\tilde{g}_{\mu\nu}$, the string-frame action given by equation (4.3.3) is transformed to the Einstein-frame action [11, 50],

$$S_E = \int d^4x\sqrt{-g}\left[\frac{1}{2}\tilde{R} + K(\tilde{\phi})\tilde{X} + L(\tilde{\phi})\tilde{X}^2 + \ldots\right], \tag{4.3.4}$$

where

$$K(\tilde{\phi}) = \frac{3}{2}\left(\frac{1}{B_g}\frac{dB_g}{d\tilde{\phi}}\right)^2 - \frac{B^{(o)}_{\tilde{\phi}}}{B_g}, \tag{4.3.5}$$

$$L(\tilde{\phi}) = 2C^{(1)}_1\alpha' B^{(1)}_{\tilde{\phi}}(\tilde{\phi})\,. \tag{4.3.6}$$

Hence this induces a Lagrangian with noncanonical kinetic terms

$$\tilde{\mathcal{L}}(\tilde{\phi}, \tilde{X}) = K(\tilde{\phi})\tilde{X} + L(\tilde{\phi})\tilde{X}^2, \tag{4.3.7}$$

where

$$\tilde{X} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\,. \tag{4.3.8}$$

In the homogeneous model given by equation (4.2.2) of the universe $\tilde{X}$ reduces to

$$\tilde{X} = \frac{1}{2}\dot{\tilde{\phi}}^2\,, \tag{4.3.9}$$

as $\tilde{\phi}(x,t) = \tilde{\phi}(t)$ due to homogeneity. Using redefination of scalar field ϕ as

$$\phi = \int d\tilde{\phi}\sqrt{\frac{|L(\tilde{\phi})|}{K(\tilde{\phi})}}\,, \tag{4.3.10}$$

where $X = \frac{|L|}{K}\tilde{X}$ and $f(\phi) = \frac{K^2}{|L|}$.

The action for the new scalar field ϕ becomes

$$S = \int d^4x\sqrt{-g}f(\phi)\mathcal{L}(X)\,, \tag{4.3.11}$$

where $\mathcal{L}(X) = X - X^2$.

Here $L < 0$ and

$$X = \frac{1}{2}g^{ij}\partial_i\phi\partial_j\phi\,. \tag{4.3.12}$$

The energy momentum components T_{ij} are obtained by taking variation of the action given by equation (4.3.11) with respect to g^{ij} as

$$\begin{aligned} T_{ij} &= \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{ij}} \\ &= f(\phi)\left[\frac{d\mathcal{L}}{dX}\partial_i\phi\partial_j\phi - g_{ij}\mathcal{L}\right]\,. \end{aligned} \tag{4.3.13}$$

These equations yield energy density

$$\begin{aligned} \rho &= f(\phi)\left[\dot{\phi}^2\frac{d\mathcal{L}}{dX} - \mathcal{L}\right] \\ &= f(\phi)\left[2X(1-2X) - (X - X^2)\right] \\ &= f(\phi)\left[X - 3X^2\right]\,, \end{aligned} \tag{4.3.14}$$

where $X = (\frac{1}{2})\dot{\phi}^2$ for homogeneous ϕ.

And energy pressure,

$$p = f(\phi)[X - X^2] \tag{4.3.15}$$

Then the equation of state of the field is given by

$$w_\phi = \frac{p}{\rho} = \frac{1-X}{1-3X}. \tag{4.3.16}$$

This shows that w_ϕ does not vary for constant X.

During radiation dominated and matter dominated era, we get $H = \dfrac{1}{2t}$ and $H = \dfrac{2}{3t}$ respectively. These two results can be written in a combined form as

$$H = \frac{2}{3(1+w_m)t}\,, \tag{4.3.17}$$

where $w_m = \dfrac{1}{3}$ for radiation and $w_m = 0$ for matter.

The conservation equation for ϕ-field and matter (including radiation) is given as

$$\dot{\rho} + \dot{\rho}_m + 3H(\rho + p + \rho_m + p_m) = 0\,, \tag{4.3.18}$$

where ρ_m and p_m are the energy and the pressure of the background matter and/or radiation respectively.

As

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0,$$

So,

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{4.3.19}$$

Using equation (4.3.17) in equation (4.3.19) we get,

$$\dot{\rho} + \frac{2}{t(1+w_m)}(1+w_\phi)\rho = 0\,. \tag{4.3.20}$$

As w_ϕ is constant for scaling solution, so from equation (4.3.16) it is found that X is also a constant and we can write energy density as $\rho = Af(\phi)$ where A is a constant.

Equation (4.3.20) yields solution

$$f(\phi) = f(\phi_o)[\frac{t_0}{t}]^{\alpha} , \tag{4.3.21}$$

where

$$\alpha = \frac{2(1+w_\phi)}{1+w_m} . \tag{4.3.22}$$

Equation (4.3.22) can be written as

$$w_\phi = \frac{(1+w_m)\alpha}{2} - 1 . \tag{4.3.23}$$

This shows that $\alpha < 2$ during matter dominance. Also for $\alpha < 0$, K-essence scalar violates the weak energy condition ($w_\phi \geq -1$). So, K-essence fluid behaves as dark energy for $1 < \alpha < 2$.

4.4 Tachyon scalar

In certain string theories, it is found that some models have Feynmann propagator with negative $(mass)^2$. The negative $(mass)^2$ is obtained when rest mass is imaginary. These ghost particles with imaginary rest mass are called tachyons. Moreover, tachyonic mode of strings attached to Dirichlet (D) branes cause D-brane instability. It is because, low energy effective action of tachyon scalar around the tachyon vacuum contains a tachyon potential which has an unstable maximum. So the scalar tachyon rolls down from the potential maximum to a stable configuration with a stable vacuum expectation value. This is the tachyon condensate. Rolling tachyon condensates have interesting cosmological consequences. It has another advantage that we already know its source, whereas in the case of quintessence, K-essence and phantom scalar is not known. A. Sen [51] showed that the decay of D-branes produces a pressureless gas with finite energy density that resembles classical dust. A rolling tachyon has an interesting equation of state whose parameter smoothly interpolates between -1 and 0 [52]. This has led to a flurry of attempts being made to construct viable cosmological models using the tachyon as a suitable candidate for the inflation at high energy [53]. The tachyon can also act as a source of dark energy depending upon the form of the tachyon potential [13, 54].

The effective Lagrangian for the tachyon on a non BPS D3-brane is described by

$$S = -\int d^4x V(\phi)\sqrt{-\det(g_{ab} + \partial_a\phi\partial_b\phi)}\,, \tag{4.4.1}$$

where $V(\phi)$ is the tachyon potential. The effective potential obtained in open string theory has the form [55]

$$V(\phi) = \frac{V_o}{\cosh(\phi/\phi_o)}\,, \tag{4.4.2}$$

where $\phi = \sqrt{2}$ for the non-BPS D-brane in the superstring and $\phi_o = 2$ for the bosonic string. There exists another type of tachyon potential which appears as the excitation of massive scalar fields on the anti-D-branes [56]. In this case the potential is given by $V(\phi) = V_0 e^{\frac{1}{2}m^2\phi^2}$ and it has a maximum at $\phi = 0$.

The energy momentum tensor components $T_{\mu\nu}$ for ϕ are obtained by taking variation of the action given by equation (4.4.1) with respect to g_{ab} as

$$T_{\mu\nu} = \frac{V(\phi)\partial_\mu\partial_\nu\phi}{\sqrt{1 + g^{ab}\partial_a\phi\,\partial_b\phi}} - g_{\mu\nu}V(\phi)\sqrt{1 + g^{ab}\partial_a\phi\,\partial_b\phi}\,. \tag{4.4.3}$$

In a spatially flat FRW background the energy density ρ and the pressure density p are given by

$$\rho = T^0_0 = \frac{V(\phi)}{\sqrt{1-\dot{\phi}^2}}\,, \tag{4.4.4}$$

$$p = -T^i_i = -V(\phi)\sqrt{1-\dot{\phi}^2}\,. \tag{4.4.5}$$

Friedmann equations for tachyon take the form

$$H^2 = \frac{8\pi G V(\phi)}{3\sqrt{1-\dot{\phi}^2}}, \tag{4.4.6}$$

and field equation for ϕ is obtained as

$$\frac{\ddot{\phi}}{1-\dot{\phi}^2} + 3H\dot{\phi} + \frac{1}{V}\frac{dV}{d\phi} = 0. \tag{4.4.7}$$

Combining these equations give

$$\frac{\ddot{a}}{a} = \frac{8\pi G V(\phi)}{3\sqrt{1-\dot{\phi}^2}}(1-\frac{3}{2}\dot{\phi}^2). \tag{4.4.8}$$

Hence an accelerated expansion occurs for $\dot{\phi}^2 < \dfrac{2}{3}$.

Equations (4.4.4) and (4.4.5) yield Raychoudhuri's equation

$$\begin{aligned}\dot{H} &= -\frac{8\pi G}{2}(\rho + p)\\ &= -\frac{8\pi G}{2}\frac{V(\phi)}{\sqrt{1-\dot{\phi}^2}}\dot{\phi}^2\,.\end{aligned} \tag{4.4.9}$$

The equation of state parameter (w_ϕ) of the tachyon is given by

$$w_\phi = \frac{p}{\rho} = \dot{\phi}^2 - 1. \tag{4.4.10}$$

The tachyon is very different from the standard field case. Irrespective of the steepness of the tachyon potential, the equation of state varies between 0 and -1, in which case the tachyon energy density behaves as $\rho \propto a^{-m}$ with $0 < m < 3$.

From Equations (4.4.7) and (4.4.9), we can express $V(\phi)$ and ϕ in terms of H and $\dot{H}$, as

$$V = \frac{3H^2}{8\pi G}\left(1 + \frac{2\dot{H}}{3H^2}\right)^{\frac{1}{2}}, \tag{4.4.11}$$

$$\phi = \int \left(-\frac{2\dot{H}}{3H^2}\right)^{\frac{1}{2}} dt. \tag{4.4.12}$$

Then the tachyon potential giving the power-law expansion, $a \propto t^p$, is

$$V(\phi) = \frac{2p}{4\pi G}\left(1 - \frac{2}{3p}\right)^{\frac{1}{2}} \phi^{-2}. \tag{4.4.13}$$

In this case the evolution of the tachyon is given by $\phi = \sqrt{\frac{2}{3}pt}$. Tachyon potentials which are not steep compared to $V(\phi) \propto \phi^{-2}$ lead to an accelerated expansion.

4.5 Phantom scalar

On the basis of observational data, in 2002, R.R.Caldwell [14] noted that equation of state parameter w has a very narrow range around $w = -1$ with more likelihood to the side of $w < -1$. So, he argued that this possibility could not be ignored for the dark energy fluid. The scalar, satisfying the condition $w < -1$, is known as phantom. This name was given by Caldwell. It is contrary to quintessence, as for phantom $(1+w) < 0$. Thus, phantom violates cosmic weak energy condition (WEC) too, whereas quintessence violates only strong energy condition. Specific models in braneworlds or Brans-Dicke scalar-tensor gravity can lead to phantom energy [57, 58]. Meanwhile the simplest explanation for the phantom dark energy is provided by a scalar field with a negative kinetic energy [14].

Historically, phantom fields were first introduced in Hoyle's version of the steady state theory. In adherence to the perfect cosmological principle, a creation field (c-field) was introduced by hoyle to reconcile the model with the homogeneous density of the universe by the creation of new matter in the voids caused by the expansion of the universe. It was further refined and reformulated in the Hoyle and Narlikar theory of gravitation.

The action of the phantom field minimally coupled to gravity is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}(\nabla\phi)^2 - V(\phi)\right], \tag{4.5.1}$$

where the sign of the kinetic term is opposite copared to the action given by equation (4.2.1) for an orinary scalar field. The energy density and isotropic pressure for a spatially flat and homogeneous scalar field $\phi(t)$ are

$$\rho = -T^0_0 = -\frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{4.5.2}$$

$$p = -\frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{4.5.3}$$

So, the equation of state (EOS) parameter for $\phi(t)$ is obtained as

$$w_\phi = \frac{p}{\rho} = \frac{\dot{\phi}^2 + 2V(\phi)}{\dot{\phi}^2 - 2V(\phi)}. \tag{4.5.4}$$

Using $w_\phi < -1$ in quation (4.5.4), it is obtained that $\dot{\phi}^2 < 0$.
It means that kinetic energy for phantom scalar is negative. So, it acts as a ghost field.

The energy conservation equation for 'ϕ' is obtained as

$$\dot{\rho} - 3H|1 + w_\phi|\rho = 0\,. \tag{4.5.5}$$

as

$$|(1 + w_\phi)| = -(1 + w_\phi)\,.$$

Equation (4.5.5) integrates to

$$\rho = Aa^{3|(1+w_\phi)|}, \tag{4.5.6}$$

where A is an integration constant. This equation shows that for a phantom scalar, energy density increasing with growing scale factor $a(t)$.

Using the equation (4.5.6), Friedmann equation is obtained as

$$H^2 = (\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3} A a^{3|(1+w_\phi)|}\,, \tag{4.5.7}$$

yielding the solution

$$a(t) = \left[\frac{3|(1 + w_\phi)|}{2}\sqrt{\frac{8\pi GA}{3}}(t_{br} - t)\right]^{\frac{-2}{3|(1+w_\phi)|}}\,, \tag{4.5.8}$$

where the subscript 'br' is for big-rip and

$$t_{br} = \frac{2}{3|(1 + w_\phi)|}\sqrt{\frac{3}{8\pi GA}}\, a_0^{-\frac{3|(1+w_\phi)|}{2}} + t_0\,. \tag{4.5.9}$$

Equations (4.5.4), (4.5.6) and (4.5.8) imply that as $t \longrightarrow t_{br}$, $a \longrightarrow \infty$, and $p \longrightarrow \infty$. It manifests big-rip singularity at $t = t_0$.

The curvature of the universe grows toward infinity within a finite time in the universe dominated by a phantom fluid. In the case of a phantom scalar field this big-rip singularity may be avoided if the potential has a maximum, e.g.,

$$V(\phi) = V_0 \left[\cosh(\frac{\alpha\phi}{m_{pl}})\right]^{-1}. \tag{4.5.10}$$

where α is constant [59]. Due to its peculiar properties, the phantom field evolves towards the top of the potential and crosses over to the other side. It turns back to execute a period of damped oscillations about the maximum of the potential at $\phi = 0$. After a certain period of time the motion ceases and the field settles at the top of the potential to mimic the de-Sitter like behavior ($w_\phi = -1$). This behavior is generic if the potential has a maximum [60]. In the case of exponential potentials the system approaches a constant equation of state with $w_\phi < -1$ [61].

4.6 Chaplygin gas model

In order to explain the mysterious phenomenon, that the universe is undergoing an accelerating expansion, one usually assumes the existence of an exotic energy with negative pressure, named dark energy, which dominates the total energy density and causes an accelerating expansion of our universe at late times. So far we have discussed a number of scalar-field models of dark energy. There exist another interesting class of dark energy models involving a fluid known as a Chaplygin gas. This fluid also leads to the acceleration of the universe at late times.

In 2001 Kamenshchik, U. Moschella and V. Pasquier [16] proposed this interesting model for dark energy, named the Chaplygin gas (CG), for which the equation of state has the form

$$p = -\frac{A}{c}, \tag{4.6.1}$$

where A is a positive constant. We recall that $p = -\dfrac{V^2(\phi)}{\rho}$ for the tachyon from equations (4.4.4) and (4.4.5). Hence the Chaplygin gas can be regarded as a special case of a tachyon with a constant potential.

S. Chaplygin introduced this equation of state as a suitable mathematical approximation for calculating the lifting force on a wing of an airplane in aerodynamics. The convenience of the Chaplygin gas is connected with the fact that the corresponding Euler equations have a very large group of symmetry, which implies their integrability. The relevant symmetry group has been recently described in modern terms [62].

The negative pressure following from the Chaplygin equation of state could also be used for the description of certain effects in deformabe solides, of strip states in the context of the quantum Hall effect and of order phenomena.

A remarkable feature of the Chaplygin gas is that, it has positive and bounded squared sound velocity

$$v_s^2 = \frac{\partial p}{\partial \rho} = \frac{A}{\rho^2}, \tag{4.6.2}$$

which is a non-trivial fact for fluids with negative pressure.

Indeed, equation (4.6.1) has a nice connection with string theory and it can be obtained from the Nambu-Goto action for d-branes moving in a $(d+2)$-dimensional space time in the light-cone parameterization. Also the

Chaplygin gas is the only fluid which, so far, admits a supersymmetric generalization. An "anti chaplygin" state equation, i.e. equation (4.6.1) with negative constant A, arises in the description of wiggly strings [63]. One of its most remarkable properties is that it describes a transition from a decelerated cosmological expansion to a stage of cosmic acceleration. The inhomogeneous Chaplygin gas can do more: it is able to combine the roles of dark energy and dark matter [64].

Later M.C. Bento, O. Bertolami and A.A. Sen [65] generalized the Chaplygin gas model by adding a new parameter in the equation of state

$$p = -\frac{A}{\rho^{\alpha}}, \quad 0 < \alpha \leq 1\,. \tag{4.6.3}$$

Here α is also a constant. This generalized model is called the Generaliged Chaplygin Gas (GCG).

The cosmological models of the Chaplygin class have at least three significant features: they describe a smooth transition from a decelerated expansion of the universe to the present epoch of cosmic acceleration, they attempt to give a unified macroscopic phenomenological description of dark energy and dark matter, and finally, they represent, perhaps, the simplest deformation of ΛCDM models.

Taking into account these attractive features, it is important to try to explain what could be the Chaplygin gas in our universe. An interesting attempt makes use of a field theory approach to the description of a (3+1)-dimensional brane immersed in a (4+1)-dimensional bulk [66]. Phenomenologically, the Chaplygin gas manifests itself as the effect of the immersion of our four-dimensional world into some multidimensional bulk.

Using the equation of state equation (4.6.1) in the conservation equation (1.2.15), we have

$$\dot{\rho} + 3H(\rho - \frac{A}{\rho}) = 0, \tag{4.6.4}$$

which integrates to

$$\rho = \sqrt{A + \frac{B}{a^6}}. \tag{4.6.5}$$

where B is a integration constant. By choosing a positive value for B we see that for small a (i.e. $a^6 \ll \frac{B}{A}$) the expression given by equation (4.6.5) is approximated by

$$\rho \sim \frac{\sqrt{B}}{a^3}\,, \tag{4.6.6}$$

which corresponds to a universe dominated by dust-like matter. For large values of the cosmological radius a, it follows that

$$\rho \sim \sqrt{A}, \quad p \sim -\sqrt{A}\,, \tag{4.6.7}$$

which, in turn, corresponds to an empty universe with a cosmological constant $\sqrt{A}$ (i.e. a de-Sitter universe). In the flat case it is also possible to find the exact solution as

$$t = \frac{1}{6\sqrt[4]{A}}\left[\ln\frac{\sqrt[4]{A+\frac{B}{a^6}}+\sqrt[4]{A}}{\sqrt[4]{A+\frac{B}{a^6}}-\sqrt[4]{A}} - 2\arctan\sqrt[4]{1+\frac{B}{Aa^6}} + \pi\right]. \tag{4.6.8}$$

One can obtain a corresponding potential for the Chaplygin gas by treating it as an ordinary scalar field. If a scalar ϕ constitute a fluid, behaving like Chaplygin gas

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,,$$
$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,.$$

So,

$$\rho + p = \dot{\phi}^2. \tag{4.6.9}$$

Incorporating this result, Raychoudhuri equation is obtained as

$$\dot{H} = -4\pi G \dot{\phi}^2. \tag{4.6.10}$$

In this case of Chaplygin gas, we have

$$H^2 = \frac{8\pi G}{3}\sqrt{A + \frac{B}{a^6}}. \tag{4.6.11}$$

Differentiation of equation (4.6.11) with respect to time t yields

$$\dot{H} = -4\pi G \frac{B}{a^6\sqrt{A + \frac{B}{a^6}}}. \tag{4.6.12}$$

Comparing equations (4.6.10) and (4.6.12), it is obtained that

$$\dot{\phi}^2 = \frac{B}{a^6\sqrt{A + \frac{B}{a^6}}}. \tag{4.6.13}$$

Also

$$\dot{\phi} = a\frac{d\phi}{da}\frac{\dot{a}}{a} = aH\frac{d\phi}{da}, \tag{4.6.14}$$

and

$$V = \frac{1}{2}\left[\sqrt{A + \frac{B}{a^6}} + \frac{A}{\sqrt{A + \frac{B}{a^6}}}\right]. \tag{4.6.15}$$

So, using equations (4.6.11) and (4.6.13) in this equation, we obtain

$$\left[\sqrt{\frac{8\pi G}{3}\sqrt{A + \frac{B}{a^6}}}\right]\frac{d\phi}{da} = \sqrt{\frac{B}{a^8\sqrt{A + \frac{B}{a^6}}}}. \tag{4.6.16}$$

This is simplified as

$$\sqrt{\frac{8\pi G}{3}}\frac{d\phi}{da} = \frac{\sqrt{B}}{a^4\sqrt{A+\frac{B}{a^6}}} \ . \tag{4.6.17}$$

Integration of equation (4.6.17) leads to

$$2\sqrt{6\pi G}\phi = -\ln\left[\frac{\sqrt{B}}{a^3} + \sqrt{\frac{A+B}{a^6}}\right] \ , \tag{4.6.18}$$

which can be written as

$$a^6 = \frac{4Be^{2\sqrt{6\pi G}\phi}}{\left[1 - Ae^{4\sqrt{6\pi G}\phi}\right]^2} \ . \tag{4.6.19}$$

Substituting this for equation (4.6.15), we obtain the potential

$$V(\phi) = \frac{\sqrt{A}}{2}\left[\cosh\sqrt{6\pi G}\phi + \frac{1}{\cosh\sqrt{6\pi G}\phi}\right] \ . \tag{4.6.20}$$

Hence, a minimally coupled field with this potential is equivalent to the Chaplygin gas model.

From equations (4.6.11) and (4.6.12), we obtained

$$\begin{aligned}
\frac{\ddot{a}}{a} &= H^2 - 4\pi G\frac{B}{a^6\sqrt{A+\frac{B}{a^6}}} \\
&= \frac{8\pi G}{3}\sqrt{A+\frac{B}{a^6}} - 4\pi G\frac{B}{a^6\sqrt{A+\frac{B}{a^6}}} \\
&= \frac{8\pi G}{3\sqrt{A+\frac{B}{a^6}}}\left[A - \frac{B}{2a^6}\right] \ .
\end{aligned} \tag{4.6.21}$$

Observations show acceleration at small red-shift having conclusive evidence from 16 Type supernova data. It shows that transition from deceleration to acceleration took place very recently in the past. So, if the transition time was t_{tr} with corresponding scale factor a_{tr} and $B = 2Aa_{tr}^6$, $\ddot{a} = 0$ at transition. Thus, from equation (4.6.21),

$$\frac{\ddot{a}}{a} = \frac{8\pi G}{3}\sqrt{\frac{A}{1 + 2(\frac{a_{tr}}{a})^6}}\left[1 - (\frac{a_{tr}}{a})^6\right]. \tag{4.6.22}$$

It leads to a very interesting conclusion that universe decelerated before time t_{tr} and accelerated after this time, which is consistent with observations.

Chapter 5

The fate of a Dark Energy Universe - Future Singularities

5.1 The future of our universe in the presence of Λ

The conventional viewpoint concerning the future of our universe is one of the following:

> (i) If the universe is spatially open or flat then it will expand for ever, alternatively
>
> (ii) If the universe is spatially closed then its expansion will be followed by recollapse.

Both these scenario's are based on the premise that the universe contains 'normal' matter, one that satisfies the strong energy condition (SEC)

$\rho + 3p \geq 0$. A cosmological Λ-term violates the SEC and it is easy to see that in its presence postulates (i) and (ii) become:

[A]. A spatially open or flat universe ($k = 0, -1$) will recollapse if the Λ-term is constant and negative;

[B]. A closed universe ($k = +1$) with a constant positive Λ-term can, under certain circumtances, expand forever.

If the Λ-term decays faster than a^{-2} then, in the case of [B], the universe will eventually recollapse if $k = +1$, alternatively if $k = 0$ then the universe may still recollapse in some regions due to the influence of local inhomogeneities in the distribution of matter. In the case of a Λ-term based on a scalar field Lagrangian, the decay of Λ is linked to the decrease of $V(\phi)$ with time. Existing observational data seem to indicate that Λ is decreasing rather slowly, if at all. For instance if one assumes $p_{\Lambda} = w\rho_{\Lambda}$, with $w = constant$, then $w < -0.6$ is indicated [67, 68] by T.D. Saini, S. Raychaudhury, V. Sahni, and A. Starobinsky, (1999) and G. Efstathiou, (1999). Since $\rho_{\Lambda} \propto a^{-3(1+w)}$ this corresponds to ρ_{Λ} decreasing less rapidly than $a^{-1.2}$ at present. However, this behaviour might change in the future.

If the Λ-term is really a cosmological constant, for $\Lambda > 0$ the expansion of the universe will rapidly approach the de-Sitter value $H = H_{\alpha} = \sqrt{\Lambda/3} = H_0\sqrt{1 - \Omega_m}$, even as the density of matter asymptotically declines to zero, $\rho_m \propto a^{-3} \longrightarrow 0$. Density perturbations will freeze to a constant value $\delta\rho/\rho \longrightarrow constant$ if they are still in the linear regime and angular anisotropies in the CMB ($\triangle T/T_l$), particularly the quadrupole, will also freeze to some constant value. However the large scale acceleration of the universe will not affect gravitationally bound systems on present scales of $R < 10h^{-1}Mpc$ (which includes our own galaxy as well as galaxy clusters). As a result the universe will soon consist of islands of matter immersed in an accelerating sea of vacuum energy: Λ.

An interesting issue in a Λ driven universe concerns how the redshift of a given object changes with time. The present redshift of an object $z \equiv z(t_0)$ located at a co-ordinate distance r is given by

$$1 + z = \frac{a(\eta_0)}{a(\eta_{em})}, \qquad \eta_{em} = \eta_0 - r\,, \tag{5.1.1}$$

$\eta_{em} = \eta(t_{em})$ is the moment when the object emitted the light which we are observing now. The physical distance to such an object is $R = ar$. Differentiating equation (5.1.1) with respect to t_0, to get $\dot{z}$, the time rate of change of the redshift of the object. It is easy to see that if $\Lambda = 0$, then $\dot{z} < 0$, for every z. In fact $z(t)$ monotonically decreases with time tending to 0 as $t \longrightarrow \infty$. On the other hand , if $\Lambda > 0, z(t)$ stops decreasing at some instant of time and then begins to increase due to the acceleration of the universe. As a result $\dot{z} > 0$ if $z < z_c$. For a flat universe the present value of z_c for which $\dot{z}_c(t_0) = 0$ can be determined from the equation

$$(1 + z_c)\left(\Omega_m + \frac{1 - \Omega_m}{(1 + z_c)^3}\right) = 1\,. \tag{5.1.2}$$

Substituting $\Omega_m = 0.3$ we get $z_c = 2.09$. (As z_c decreases with increasing Ω_m.) Thus we find that there is a reversal in the sign of $\dot{z}$ for sufficiently close objects, an effect that may even be observable in the not-too-distant future !

Another interestig aspect of a Λ-dominated universe is that, provided the universe is accelerating over a large enough region, the local neighborhood of an observer from which the observer is able to receive signals will eventually contract and shrink. As a result even those regions of the universe which are observable to us at present will eventually be hidden from view. Conversely, the co-ordinate volume of space which may be affected by our civilization in the infinite future is finite and its boundary is given by $r_H = \eta(t = \infty) - \eta_0$. These features are related to the appearance of a de-Sitter-like future event horizon in a universe which inflates in the future. This reasoning applies to galaxies and other high redshift objects which we are currently observing: if the redshift of an object is $z > z_H$ then such an object will remain 'out of bounds' to our civilization forever ! The redshift z_H can be determined from the following considerations. Light propogation in a FRW universe is described by setting $ds^2 = 0$ in equation (1.2.1). Considering an event at (r_1, t_1) and observer location at $r = 0$, we get

$$\int_0^{r_1} \frac{dr}{\sqrt{1 - kr^2}} = \int_{t_1}^{t_2} \frac{dt'}{a(t')} \ . \tag{5.1.3}$$

If the integral in the R.H.S. of equation (5.1.3) diverges (as $t \longrightarrow \infty$) then an observer at $r = 0$ will be able to receive signals from any event. For $a \propto t^p$, this implies $p < 0$, that is the universe is decelerating. In an accelerating universe the integral converges which means that one can only receive signals from those events which satisfy

$$\int_0^{r_1} \frac{dr}{\sqrt{1 - kr^2}} \leq \int_{t_1}^{\infty} \frac{dt'}{a(t')}. \tag{5.1.4}$$

Taking the equality sign and assuming the expansion is de Sitter-like, so that $a \propto \exp Ht, H = \sqrt{\Lambda/3}$, we get $r_1 \equiv r_H, R = a_0 r_H = H^{-1}$, which is the proper distance to the event horizon in de-Sitter space. Next considering an event at (r_H, t_1) whose signals are being received by us today $(t = t_0)$ so that $\frac{a_0}{a(t)} - 1 = z_H$ is its redshift. Substituting $\frac{dz}{dt} = -(1+z)H(z)$ in equation (5.1.3) we obtain (assuming $k = 0$)

$$H_0 a_0 r_H = \int_0^{z_H} \frac{dz}{[\Omega_\Lambda + \Omega_\Lambda(1+z)^3]^{\frac{1}{2}}} \,. \tag{5.1.5}$$

Now assuming that $r_1 = r_H$ is arbitrarily close to the horizon so that light emitted today $(t_1 = t_0)$ will reach us at $t \longrightarrow \infty$, from equation (5.1.4) we obtain

$$H_0 a_0 r_H = \int_{-1}^{0} \frac{dz}{[\Omega_\Lambda + \Omega_\Lambda(1+z)^3]^{\frac{1}{2}}} \,. \tag{5.1.6}$$

Equating equations (5.1.5) and (5.1.6) we get

$$\int_1^{1+z_H} \frac{dx}{\sqrt{1 - \Omega_m + \Omega_m x^3}} = \int_0^1 \frac{dx}{\sqrt{1 - \Omega_m + \Omega_m x^3}}. \tag{5.1.7}$$

which can be solved for z_H. If $\Omega_m = 0.3$, then $z_H = 1.8$, a rather small redshift since we see many galaxies and QSO's at much higher redshifts ($z_H \longrightarrow \infty$ as $\Omega_m \longrightarrow 1$). For $\Lambda = constant, z < z_H$ is the 'sphere of influence of our civilization' since celestial objects with $z > z_H$ will always remain inaccessible to signals emitted by us either now or in the future. Thus comoving observers once visible to us will gradually disappear from view as light emitted by them gets redshifted and declines in intensity.

5.2 Classification of future singularities

In this section we shall discuss the future singularities which can in principle appear in a dark energy universe. When the equation of state of dark eneregy is less than -1, the universe reaches a Big Rip singularity within a finite time.

In this case the null condition $\rho + p \geq 0$, is violated. In 2004 J.D.Barrow [69] showed that a different type of future singularity can appear at a finite time even when the strong energy condition

$$\rho + 3p \geq 0, \qquad \rho + p \geq 0\,, \tag{5.2.1}$$

is satisfied. This sudden future singularity corresponds to the one in which the pressure density p diverges at $t = t_s$ but the energy density ρ and the scale factor a are finite.

There exist a number of different finite-time singularities in a dark energy universe. According to S.Nojiri, S.D.Odintsov and S.Tsujikawa, the future-singularities can be classified into the following four classes [70]

(i) Type I ("Big Rip"): For $t \longrightarrow t_s,\ a \longrightarrow \infty,\ \ \rho \longrightarrow \infty$ and $|p| \longrightarrow \infty$.

(ii) Type II ("Sudden"): For $t \longrightarrow t_s,\ a \longrightarrow a_s,\ \ \rho \longrightarrow \rho_s$ and $|p| \longrightarrow \infty$.

(iii) Type III: For $t \longrightarrow t_s,\ a \longrightarrow a_s,\ \ \rho \longrightarrow \infty$ and $|p| \longrightarrow \infty$.

(iv) Type IV: For $t \longrightarrow t_s, a \longrightarrow a_s, \rho \longrightarrow 0, |p| \longrightarrow 0$ and higher derivatives of H diverge.

Here t_s, a_s and ρ_s are constants with $a_s \neq 0$. The type I corresponds to the Big Rip singularity [71], whereas the type II corresponds to the sudden future singularity mentioned above. Type III singularity appears for the model with $p = -\rho - A\rho^{\alpha}$ and is different from the sudden future singularity in the sense that ρ diverges. The type IV singularity appears in the model with

$$p = -\rho - \frac{AB\rho^{\alpha\beta}}{A\rho^{\alpha} + B\rho^{\beta}}\,, \tag{5.2.2}$$

We consider the equation of state of dark energy which is given by

$$p = -\rho - f(\rho), \tag{5.2.3}$$

where $f(\rho)$ is a function in terms of ρ.

The function $f(\rho)$ characterizes the deviation from a ΛCDM cosmology. Nojiri and Odintsov [58] proposed the function of the form $f(\rho) \propto \rho^{\alpha}$. For the equation of state given by equation (5.2.3) with $f(\rho) \neq 0$ the continuity equation is written in an integrated form as

$$a = a_0 \exp\left(\frac{1}{3}\int \frac{d\rho}{f(\rho)}\right), \tag{5.2.4}$$

where a_0 is constant.

In the absence of any barotropic fluid other than dark energy, Friedmann equation becomes,

$$H^2 = \frac{k^2}{3}\rho\,. \tag{5.2.5}$$

where $k^2 = 8\pi G$ and it leads to other relations

$$t = \int \frac{d\rho}{k\sqrt{3\rho}f(\rho)}\ . \tag{5.2.6}$$

and

$$\begin{aligned}\frac{\ddot{a}}{a} =& -\frac{k^2}{6}(\rho + 3p)\\ =& \frac{k2}{6}\left[2\rho + 3f(p)\right]\ .\end{aligned} \tag{5.2.7}$$

5.3 Type I and III singularities

Type I and III singularities appear when the function $f(\rho)$ is of the form

$$f(\rho) = A\rho^{\alpha} , \tag{5.3.1}$$

where A and α are constants. We consider a situation in which ρ goes to infinity with positive α. From equations (5.2.4) and (5.2.6), we obtain

$$a = a_0 \exp\left[\frac{\rho^{1-\alpha}}{3(1-\alpha)A}\right] , \tag{5.3.2}$$

and

$$t = t_s + \frac{2}{\sqrt{3}kA}\frac{\rho^{-\alpha+\frac{1}{2}}}{1-2\alpha} \,, \text{for} \;\; \alpha \neq \frac{1}{2} \,, \tag{5.3.3}$$

$$t = t_s + \frac{\ln\rho}{\sqrt{3}kA} \;, \;\; \text{for} \;\; \alpha = \frac{1}{2} \,, \tag{5.3.4}$$

where t_s is constant.

When $\alpha > 1$, the scale factor a is finite even for $\rho \longrightarrow \infty$.

When $\alpha < 1$ we find $a \longrightarrow \infty (a \longrightarrow 0)$ as $\rho \longrightarrow \infty$ for $A > 0 (A < 0)$.

If $\alpha > \frac{1}{2}$ the energy density ρ diverges in the finite future or past ($t = t_s$). On the other hand , if $\alpha \leq \frac{1}{2}$, ρ diverges in the infinite future or past. Since the pressure is given by

$$p \sim -\rho - Ap^{\alpha} \,, \tag{5.3.5}$$

p always diverges when ρ become infinite. The equation of state of dark energy is

$$w = \frac{p}{\rho} = -1 - A\rho^{\alpha-1} \,. \tag{5.3.6}$$

when $\alpha > 1$, we have $w \longrightarrow +\infty (w \longrightarrow -\infty)$ as $\rho \longrightarrow \infty$ for $A < 0 (A > 0)$. Mean while when $\alpha < 1$, $w \longrightarrow -1 + 0 (-1 - 0)$ for $A < 0 (A > 0)$.

From above discussion, we can classify the singularities as follows:

(a) $\alpha > 1$,

There exists a type III singularity, $w \longrightarrow +\infty(-\infty)$ if $A < 0(A > 0)$.

(b) $\frac{1}{2} < \alpha < 1$,

There is a type I future singularity for $A > 0$. When $A < 0$, one has $a \longrightarrow 0$ as $\rho \longrightarrow \infty$. Hence if the singularity exists in the past (future), we may call it Big Bang (Big Crunch) singularity. $w \longrightarrow -1+0(-1-0)$ if $A < 0(A > 0)$.

(c) $0 < \alpha \leq \frac{1}{2}$, There is no finite future singularity.

5.4 Type II singularity

Type II singularity appears when $f(\rho)$ is of the form

$$f(\rho) = C(\rho_0 - \rho)^{-\gamma}\,, \tag{5.4.1}$$

where C, ρ_0 and γ are constants with $\gamma > 0$.

We study the case in which ρ is smaller than ρ_0. In the limit $\rho \longrightarrow \rho_0$, the pressure p becomes infinite because of the divergence of $f(\rho)$. The scalar curvature R also diverges since $R = 2k^2(\rho - 3p)$. The equation of state of dark energy is

$$w = -1 - \frac{C}{\rho(\rho_0 - \rho)^{\gamma}}\,. \tag{5.4.2}$$

Hence, $w \longrightarrow -\infty$ for $C > 0$ and $w \longrightarrow \infty$ for $C < 0$ as $\rho \longrightarrow \rho_0$.

From equation (5.2.4) the scale factor is given by

$$a = a_0 \exp\left[-\frac{(\rho_0 - \rho)^{\gamma+1}}{3C(\gamma + 1)}\right], \tag{5.4.3}$$

which means that a is finite for $\rho = \rho_0$. Since the Hubble rate $H \propto \sqrt{\rho}$ is nonsingular, $\dot{a}$ remains finite. On the other hand equation (1.2.17) shows that $\ddot{a}$ diverges for $\rho \longrightarrow \rho_0$.

By using equation (5.2.6) we obtained the relation around $\rho \longrightarrow \rho_0$ as

$$t \simeq t_s - \frac{(\rho_0 - \rho)^{\gamma+1}}{kC\sqrt{3\rho_0}(\gamma+1)} \ , \tag{5.4.4}$$

where t_s is an integration constant. Then we have $t = t_s$ for $\rho = \rho_0$. The above discussion shows that the function $f(\rho)$ in equation (5.4.1) gives rise to the Type II singularity. We note that the strong energy condition given by equation (5.2.1) is satisfied for $C < 0$ around $\rho = \rho_0$. This means that the sudden singularity appears even in the case of a non-phantom dark energy ($w > -1$).

This type II singularity always appears when the denominator of $f(\rho)$ vanishes at a finite value of ρ. The model given by equation (5.3.1) with negative α is a special case of the model given by equation (5.4.1) with $\rho_0 = 0$.

5.5 Type IV singularity

In 2005, three authors S.Nojiri, S.D.Odintsov and S.Tsujikawa [70] shown that the type IV singularity can appear in the model given by

$$f(\rho) = \frac{AB\rho^{\alpha+\beta}}{A\rho^{\alpha} + B\rho^{\beta}} \ , \tag{5.5.1}$$

where A, B, α and β are constants. This model also gives rise to the type I, II, III singularities.[70]

In what follows we shall study the case with $\alpha = 2\beta - 1$. Then equation's (5.2.4) and (5.2.6) are integrated to give

$$a = a_0 \exp\left[-\frac{1}{3}\left\{\frac{\rho^{-\alpha+1}}{(\alpha-1)A} + \frac{\rho^{-\beta+1}}{(\beta-1)B}\right\}\right], \tag{5.5.2}$$

and

$$\begin{aligned}\frac{2}{4\beta-3}\rho^{-\frac{(4\beta-3)}{2}} + \frac{2A}{(2\beta-1)B}\rho^{-\frac{(2\beta-1)}{2}} &= -\sqrt{3}\kappa A(t-t_s) \\ &\equiv \tau, \end{aligned} \tag{5.5.3}$$

where t_s is an integration constant and equation (5.5.3) is valid for $\beta \neq 1$, $\beta \neq \frac{3}{4}$ and $\beta \neq \frac{1}{2}$.

Now , we consider the $0 < \beta < \dfrac{1}{2}$. In this case the pressure density behaves as

$$p = -\rho - B\rho^{\beta}, \text{ when } \quad \rho \sim 0 \tag{5.5.4}$$

Equation (5.5.3) shows that $t \longrightarrow t_s$ as $\rho \longrightarrow \rho_0$. Then from equation (5.5.4) we obtain $p \longrightarrow 0$ and $\rho \longrightarrow 0$ as $t \longrightarrow t_s$.

By using equations (5.5.2) and (5.5.3) we obtain the following relation around $t = t_s$,

$$ln(\frac{a}{a_0}) \propto \tau^s, \qquad s = 1 - \frac{1}{2\beta - 1}, \tag{5.5.5}$$

Hence the scale factor is finite ($a = a_0$) at $t = t_s$.

From equation (5.5.5) , we find that $s > 2$ for $0 < \beta < \frac{1}{2}$ which means that H and $\dot{H}$ are finite. However $\dfrac{d^n H}{dt^n}$ diverges for $n > -\dfrac{1}{(2\beta-1)}$ as long as s is not an integer . This corresponds to the type IV singularity in which higher-order derivatives of H exhibit divergence even if a, ρ and p are finite as $t \longrightarrow t_s$. In this case, $w \to +\infty(-\infty)$ for $B < 0(B > 0)$.

5.6 A possible avoidance of Big-smash

Phantom dominated universe encounters big-smash problem in future. To avoid this problem, one way is to throw the idea of phantom out. But it can not be done due to experimantal support to it. Another way is to investigate a phantom model, where this problem does not arise. In this direction, first attempt was made by Elizalde et. al. These authors suggested that making quantum corrections in energy density and pressure near big-rip singularity, this problem can be avoided. Later on 2005, a possible escape from this problem was suggested by S.K.Srivastava [72]. In this model, it was suggested that well-behaved scale factor $a(t)$ could be obtained if phantom fluid behaves as generalized Chaplygin gas as well as barotropic fluid simultaneously.

The generalized Chaplygin gas (GCG) is characterized by equation of state

$$p = -\frac{A}{\rho^{\frac{1}{\alpha}}} \,, \tag{5.6.1}$$

where $1 \leq \alpha < \infty$. Equation of state for the barotropic fluid is given as

$$p = w\rho \quad \text{where} \quad w < -1. \tag{5.6.2}$$

The conservation equation for GCG is

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) \,, \tag{5.6.3}$$

and integrating, it is obtained that

$$\rho^{\frac{(1+\alpha)}{\alpha}}(t) = A + \left(\rho_0^{\frac{(1+\alpha)}{\alpha}} - A\right)\left[\frac{a_0}{a(t)}\right]^{\frac{3(1+\alpha)}{\alpha}} \,, \tag{5.6.4}$$

with $\rho_0 = \rho(t_0)$ and $a_0 = a(t_0)$, where t_0 is the present time.

Equations (5.6.1) and (5.6.2) yiled w as

$$w(t) = -\frac{A}{\rho^{\frac{(1+\alpha)}{\alpha}}(t)} , \tag{5.6.5}$$

So, at $t = t_0$, it is obtained as

$$A = -w_0 \rho_0^{\frac{(1+\alpha)}{\alpha}} , \tag{5.6.6}$$

with $w_0 = w(t_0)$. From equation's (5.6.4) and (5.6.6), it is obtained that

$$\rho = \rho_0 \left[-w_0 + (1+w_0)(\frac{a_0}{a(t)})^{\frac{3(1+\alpha)}{\alpha}} \right]^{\frac{\alpha}{(1+\alpha)}} , \tag{5.6.7}$$

with $w_0 < 1$. Connecting equations (4.5.2), (5.6.1) and (5.6.6), it is obtained that

$$\dot{\phi}^2 = \frac{\rho^{\frac{(1+\alpha)}{\alpha}} + \rho_0^{\frac{(1+\alpha)}{\alpha}} \omega_0}{\rho^{\frac{1}{\alpha}}} . \tag{5.6.8}$$

Connecting equations (5.6.7) and (5.6.8), it is obtained that

$$\dot{\phi}^2 = \frac{(1+w_0)\rho_0^{\frac{(1+\alpha)}{\alpha}} (\frac{a_0}{a})^{\frac{3(1+\alpha)}{\alpha}}}{\left[-w_0 + (1+w_0)(\frac{a_0}{a})^{\frac{3(1+\alpha)}{\alpha}} \right]^{\frac{\alpha}{(1+\alpha)}}} . \tag{5.6.9}$$

This equation shows that $\dot{\phi}^2 > 0$ (giving positive kinetic energy) for $w_0 > -1$, which is the case of quintessence and $\dot{\phi}^2 < 0$ (giving negative kinetic energy) for $w_0 < -1$, being the case of super-quintessence (phantom).

Thus, it is shown that dual behaviour of dark energy fluid, obeying equations (5.6.1) and (5.6.4) is possible for scalars, frequently used for cosmological dynamics.

Now, the Friedmann equation, with dominance of dark energy having double fluid behaviour, is

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\Omega_0\left[|w_0| + (1-|w_0|)[\frac{a_0}{a(t)}]^{\frac{3(1+\alpha)}{\alpha}}\right]^{\frac{\alpha}{(1+\alpha)}} , \qquad (5.6.10)$$

where $|w_0| > 1$. H_0 is the present value of Hubble's constant and $\Omega_0 = \dfrac{\rho_0}{\rho_{cr,\,0}}$ with $\rho_{cr,\,0} = \dfrac{3H_0^2}{8\pi G_N}$ (G_N being the Newtonian gravitational constant).

Neglecting higher powers of $\dfrac{1-|w_0|}{|w_0|}\left[\dfrac{a_0}{a(t)}\right]^{\frac{3(1+\alpha)}{\alpha}}$, equation (5.6.10) is written as

$$\frac{\dot{a}}{a} \simeq H_0\sqrt{\Omega_0}|w_0|^{\frac{\alpha}{2(1+\alpha)}}\left[1 + \frac{\alpha(1-|w_0|)}{2(1+\alpha)|w_0|}\left(\frac{a_0}{a(t)}\right)^{\frac{3(1+\alpha)}{\alpha}}\right]. \qquad (5.6.11)$$

Equation (5.6.11) is integrated to

$$a(t) = \frac{a_0}{[2(1+\alpha)|w_0|]^{\frac{\alpha}{3(1+\alpha)}}}[\{\alpha + (2+\alpha)|w_0|\} e^{H_0\sqrt{\Omega_0}|w_0|^{\frac{\alpha}{2(1+\alpha)}}\frac{3(1+\alpha)}{\alpha}(t-t_0)} - \alpha(1-|\omega_0|)]\,, \qquad (5.6.12)$$

yilding accelerated expansion of the universe with $a(t) \longrightarrow \infty$ as $t \longrightarrow \infty$, consistent with observations. Moreover, equations (5.6.1), (5.6.7) and (5.6.12) show that ρ and p are finite when $t \longrightarrow \infty$. Thus, big-rip singularity is avoided.

Bibliography

[1] S. J. Perlmutter et al., Astrophys. J. **517**, 565 (1999);[astro-ph/9812133]

[2] A. G. Riess et al., Astron. J. **116**, 1009 (1998); Astron. J. **117**,707 (1999).

[3] D. N. Spergel et al. [WMAP Collaboration], Astrophys. J. Suppl. **148**, 175 (2003) [arXiv: astro-ph/0302209].

[4] P. J. Steinhardt and N. Turok, Phys. Rev. D **65**, 126003 (2002).

[5] A. G. Riess et al. [Supernova Search Team Collaboration], Astrophys. J. **607**, 665 (2004).

[6] V. Sahni and A. A. Starobinsky, Int. J. Mod. Phys. D **9**, 373 (2000); V. Sahni, Lect. Notes Phys. **653**, 141 (2004).

[7] T. Padmanabhan, Phys. Rep. **380**, 235 (2003); T. Padmanabhan, Current Science, **88**, 1057 (2005).

[8] P.J.E. Peebles and B. Ratra, Rev. Mod. Phys. **75**, 559 (2003).

[9] E. J. Copland, M. Sami, S.Tsujikawa, Int. J. Mod. Phys. D **15**, 1753 (2006)[hep-th/0603057].

[10] R. R. Caldwell, M. Kamionkowski and N. N. Weinberg, Phys. Rev. Lett. **91**,071301(2003); S. Nesseris and L. Perivolaropoulos, Phys. Rev. D **70**,123529(2004).

[11] C. Armendariz-Picon, T. Damour and V. Mukhanov, Phys. Lett. B **458**, 209 (1999); J. Garrige and V. Mukhanov, Phys. Lett. B **458**, 219 (1999).

[12] T. Chiba, T. Okabe and M. Yamaguchi, Phys. Rev. D **62**, 023511 (2000).

[13] T. Padmanabhan and T. R. Choudhury, Phys. Rev. D **66**, 081301 (2002).

[14] R. R. Caldwell, Phys. Lett. B 545, 23-29(2002).

[15] A. Sen, JHEP **9910**, 008 (1999); M. R. Garousi, Nucl. Phys. B **584**, 284 (2000); Nucl. Phys. B **647**, 117 (2002).

[16] A. Y. Kamenshchik, U. Moschella and V. Pasquier, Phys. Lett. B **511**, 265(2001).

[17] S. Nojiri and S. D. Odintsov, Phys. Rev. D **68**, 123512(2003).

[18] S. Weinberg, "Gravitation and Cosmology", John Wiley and Sons, Inc. (1972).

[19] E. W. Kolb and M. Turner, The Early Universe, Addison-Wesley (1990).

[20] A. Einstein, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1917**, 142 (1917).

[21] D. N. Spergel, R. Bean, O. Dore, M. R. Nolta, CL. Bennett, et. al., Astrophys. J. Suppl. **170**, 377-408 (2007).

[22] R. G. Crittenden, N. Turok, "Looking for Lambda with the Rees-Sciama Effect", Phys. Rev. Lett., **76**, 575 (1996).

[23] Ho et al., Phys. Rev. D. (2008).

[24] Giannantonio et al., Phys. Rev. D. (2008).

[25] B. Ratra and J. Peebles, Phys. Rev. D **37**, 321 (1988).

[26] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989).

[27] G. Volovik, Phys. Repts. **351**, 58 (2001).

[28] J. Iliopoulos, and B. Zumino, Nucl. Phys. B **76**, 310 (1974).

[29] J. E. Felten, and R. Isaacman, Rev. Mod. Phys. **58**, 689 (1986).

[30] J. M. Overduin, F. I. Cooperstock, Phys. Rev. D **58**, 043506 (1998).

[31] V. Petrosian, E. Salpeter and P. Szekeres, Astrophys. J. **147** 1222 (1967).

[32] J. P. Ostriker, P. J. Steinhardt, Nature, **377**, 600 (1995).

[33] R. A. Schommer, N. B. Suntzeff and R. C. Smith, astro-ph/9909088 (1999).

[34] S. M. Carroll, W. H. Press and E. L. Turner, Ann. Rev. Astron. Astrophys, **30**, 499 (1992).

[35] M. Fukugita, T. Futamase, M. Kasai, MNRAS, **246**, 24 (1990).

[36] E. L. Turner, APJ, **365**,L 43 (1990).

[37] S. M. Carroll, "The Cosmological Constant", Living Rev. Relativity, **4**, 1 (2001).

[38] E. E. Falco, C. S. Kochanek and J. A. Munoz, Ap. J. **494**, 47 (1998).

[39] K. H. Chae et. al., Phys. Rev. Lett. **89**, 151301 (2002).

[40] J. Maldacena and C. Nunez, Int. J. Mod. Phys., A **16**, 822 (2001).

[41] S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, Phys. Rev. D **68**, 046005 (2003).

[42] R. Bousso and J. Polchinski, JHEP **0006**, 006 (2000).

[43] S. Giddings, S. Kachru and J. Polchinski, Phys. Rev. D **66**, 106006 (2002).

[44] E. Witten, Nucl. Phys. B **474**, 343 (1996).

[45] S. Kachru, J. Pearson and H. Verlinde, JHEP **0206**, 021 (2002).

[46] J. L. Feng, J. March-Russell, S. Sethi and F. Wilczek, Nucl. Phys. B **602**, 307 (2001).

[47] J. D. Brown and C. Teitelboim, Phs. Lett. B **195**, 177 (1987); J. D. Brown and C. Teitelboim, Nucl. Phy. B **297**, 787 (1988).

[48] J. J. Van der Bij et. al., Physica A **116**, 307 (1982); W. Buchmuller and N. Dragon, Phys. Lett. B **207**, 292 (1988).

[49] C. Armendariz-Picon, V. Mukhanov and P.J. Steinhardt, Phys. Rev. Lett. **85**, 4438 (2000); C. Armendariz-Picon, V. Mukhanov and P.J. Steinhardt, Phys. Rev. D **63**, 103510 (2001).

[50] M. Gasperini and G. Veneziano, Phys. Rept. **373**, 1 (2003).

[51] A. Sen, JHEP **0204**, 048 (2002); A. Sen, JHEP **0207**, 065 (2002).

[52] G. W. Gibbons, Phys. Lett. B **537**, 1 (2002).

[53] A.Mazumdar, S. Panda and A. Perez-Lorenzana, Nucl. Phys. B **614**, 101 (2001); A.Feinstein, Phys. Rev. D **66** , 063511 (2002).

[54] E. J. Copland, M. R. Garousi, M. Sami and S. Tsujikawa, Phys. Rev. D **71** , 043003 (2005).

[55] Kutasov and V. Niarchos, Nucl. Phys. B **666**, 56 (2003).

[56] M.R. Garousi, M. Sami and S. Tsujikawa, Phys. Rev. D **70**, 043536 (2004); Phys. Lett. B **606**, 1 (2005); P. Chingangbam and T. Qureshi, Int. J. Mod. Phys. A **20**, 6083 (2005).

[57] V. Sahni and Y. Shtanov, JCAP **0311**, 014 (2003); V. Sahni and Y. Shtanov, IJMPD **67**, 1515 (2000).

[58] E. Elizalde, S. Nojiri and S. D. Odintsov, Phys. Rev. D **70**, 043539 (2004).

[59] P. Singh, M. Sami and N. Dadhich, Phys. Rev. D **68** 023522 (2003).

[60] S. M. Carroll, M. Hoffman and M. Trodden, Phys. Rev. D **68**, 023509 (2003).

[61] M. Sami and A. Toporensky, Mod. Phys. Lett. A **19**, 1509 (2004).

[62] D. Bazeia and R. Jackiw, Ann. Phys. (NY), **270**, 246 (1998).

[63] A. Vilenkin, Phys. Rev. D **41**, 3038 (1991).

[64] N. Bilic, G. B. Tupper and R. Violler, Phys. Lett. B **535**, (2002)

[65] M. C. Bento, O. Bertolami and A. A. Sen, Phys. Rev. D **66**, 043507 (2002).

[66] R. Sundrum, Phys. Rev. D **59**, 085009 (1999).

[67] G. Efstathiou, MNRAS **310** , 842 (1999).

[68] T. D. Saini, S. Raychaudhury, V. Sahni and A. Starobinsky, astro-ph/9910231 (1999).

[69] J.D. Barrow, Class. Quant. Grav. **21**, 5619 (2004).

[70] S.Nojiri, S.D. Odintsov and S. Tsujikawa, Phys. Rev. D **71**, 063004 (2005).

[71] R. R. Caldwell, M. Kamionkowski, N. N. Weinberg, Phys. Rev. Lett., **91**, 071301 (2003).

[72] S. K. Srivastava, Phys. Lett. B, **619**, 1 (2005).

Brief Bio-data

1. **Name:** BHARAT BORAH
2. **Sex:** Male
3. **Date of birth:** 1^{st} March, 1984.
4. **Father's Name:** Mr. Jitendra Nath Borah
5. **Mother's Name:** Mrs. Kusum Borah
6. **Nationality:** Indian
7. **Permanent Address:** Vill. - Kujidah,
P.O. - Hatichong
Dist. - Nagaon, Assam,
Pin - 782 142.
8. **Academic Qualification:** M. Sc. in Mathematics,
Gauhati University.
9. **Workshop/Conference attended:**
   UGC-SAP workshop on *Algebra, Algebraic Topology and related topics*, held at North-Eastern Hill University, Meghalaya, organized by Dept. of Mathematics, NEHU, Meghalaya, from March 15 to 20, 2010.