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The homomorphism $M^* \otimes N \to \mathrm{Hom}(M,N)$

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Abstract. In this note we prove two theorems. In theorem 1 we prove that if M and N are two non-zero reflexive modules of finite projective dimensions over a Gorenstein local ring, such that Hom (M, N) is a third module of syzygies, then the natural homomorphism $M^* \otimes N \to$ Hom (M, N) is an isomorphism. This extends the result in [7]. In theorem 2, we prove that projective dimension of a module M over a regular local ring R is less than or equal to n if and only if $\mathrm{Ext}^n_R(M, R) \otimes M \to \mathrm{Ext}^n_R(M, M)$ is surjective; in which case it is actually bijective. This extends the usual criterion for the projectivity of a module.

Keywords. Homomorphism; Gorenstein local ring; isomorphism; projectivity.

Throughout, R will denote a commutative noetherian local ring with identity. All modules will be assumed to be finitely generated and unitary.

Auslander and Goldman [2] in their study of orders in simple algebras over the quotient field of a regular local ring R, discovered that if M is a reflexive R-module such that $\mathrm{Hom}_R(M, M)$ is free, then M itself is free. This is generalized in [7] to assert if M and N are non-zero reflexive modules over R, a regular local ring, such that $\mathrm{Hom}_R(M, N)$ is non-zero free, then so are M and N. This is done by showing that the natural homomorphism $M^* \otimes N \to$ Hom (M, N) is an isomorphism. Here $*$ denotes duals. This can be improved further. This is theorem 1 of the present note. The projectivity of a module M is equivalent to the surjectivity of the homomorphism $M^* \otimes M \to \mathrm{Hom}(M, M)$. Theorem 2 is an extension of this giving a criterion for a module over a regular local ring to have projective dimension less than or equal to a given non-negative integer.

In what follows we shall abbreviate projective dimension as (p.d.), Krull dimension will be simply denoted as dim. All modules are zeroth syzygies. A module M is said to be an nth syzygy, for a positive integer n, if there is an exact sequence, $0 \to M \to F_{n-1} \to \ldots\ldots \to F_1 \to F_0$, where $F_0, F_1, \ldots$ are free modules. We shall use [8] as a reference for standard terms in Commutative Algebra. If $\ldots \to F_n \to F_{n-1} \ldots \to F_0 \to M \to O$ is a projective resolution of a module M, the kernal of $F_i \to F_{i-1}$ will be denoted as $\Omega^{i+1} M$, with the understanding that $F_{-1} = M$ and $\Omega^0 M = M$; the co-kernal of $F_i^* \to F_{i+1}^*$ will be denoted as $\overline{D\Omega^i M}$. Although these definitions depend on the chosen projective resolution of M, they are well defined up to 'projective equivalence' (Auslander and Bridger [1]). Finally a module N is said to be 'rigid' if whenever $\mathrm{Tor}_1^R(M, N) = (0)$, for a module M, then $\mathrm{Tor}_j^R(M, N) = (0)$ for all $j \geqslant 1$. Over a regular local ring any module is 'rigid'. This is a theorem of by Lichtenbaum [5]. We now state the main theorem of this note:

THEOREM 1 Suppose M and N are non-zero reflexive modules of finite projective dimensions over a Cohen-Macaulay local ring R and N is rigid. Then if either $\mathrm{Ext}^1_R(\mathrm{Hom}(M, N)^*, R) = (0)$, or the ring R is Gorenstein and $\mathrm{Hom}(M, N)$ is a third module of syzygies, then the natural homomorphism $M^* \otimes N \to \mathrm{Hom}(M, N)$ is an isomorphism.

Remark. Let R be regular local and M, N reflexive R-modules. If $\mathrm{Hom}(M, N)$ is a free module, then $\mathrm{Ext}^1_R(\mathrm{Hom}(M, N)^*, R) = (0)$ so that the theorem 1 above implies the proposition in [7]. If $\mathrm{Ext}^1(\mathrm{Hom}(M, M), R) = (0)$, then since $\mathrm{Hom}(M, M) \cong \mathrm{Hom}(M^*, M^*)^*$, we get $\mathrm{Ext}^1(\mathrm{Hom}(M^*, M^*)^*, R) = (0)$ so that by the theorem above, the natural homomorphism $M^{**} \otimes M^* \to \mathrm{Hom}(M^*, M^*)$ is an isomorphism showing that M^* is free and so M is free. This is theorem 2.1 in [6]. The method of proving theorem 1 will be similar to that of [6] to some extent.

Proof of theorem 1 We first claim that the given conditions mean $\mathrm{Ext}^1_R(M, N) = 0$. This is proved by induction on dim R. Firstly, if q is a prime ideal of height at most two, then M_q being a reflexive R_q-module, depth M_q = depth R_q. Then the formula, $\mathrm{P.d.}M_q$ + depth M_q = depth R_q, implies M_q is R_q-free. Hence, $\mathrm{Ext}^1_{R_q}(M_q, N_q) = (0)$. We can therefore assume $\dim R \geqslant 3$, and that $\mathrm{Ext}^1_{R_p}(M_p, N_p) = (0)$, for all non-maximal prime ideals p of R. Then either $\mathrm{Ext}^1_R(M, N) = 0$, or it is a non-zero module of finite length. Let

$$0 \to \mathrm{L} \to \mathrm{F} \to \mathrm{M} \to 0, \tag{1}$$

be an exact sequence with F free. Application of the functor Hom (, N) to (1) gives the exact sequence,

$$0 \to \mathrm{Hom}(\mathrm{M}, \mathrm{N}) \to \mathrm{Hom}(\mathrm{F}, \mathrm{N}) \to \mathrm{Hom}(\mathrm{L}, \mathrm{N}) \to \mathrm{Ext}^1(\mathrm{M}, \mathrm{N}) \to 0. \tag{2}$$

Now (2) can be broken into two short exact sequences

$$0 \to \mathrm{Hom}(\mathrm{M}, \mathrm{N}) \to \mathrm{Hom}(\mathrm{F}, \mathrm{N}) \to \mathrm{T} \to 0. \tag{3}$$

$$0 \to \mathrm{T} \to \mathrm{Hom}(\mathrm{L}, \mathrm{N}) \to \mathrm{Ext}^1(\mathrm{M}, \mathrm{N}) \to 0. \tag{4}$$

By the assumption on $\mathrm{Ext}^1(M, N)$, some power of maximal ideal of R annihilates $\mathrm{Ext}^1(M, N)$. Then by Cohen-Macaulay property of R, we find that grade $\mathrm{Ext}^1_R(M, N) = \dim R \geqslant 3$, *i.e.* $\mathrm{Ext}^j_R(\mathrm{Ext}^1_R(M, N), R) = (0)$ for $j = 0, 1, 2$. Therefore, dualizing (3) and (4), we get the following exact sequences

$$0 \to \mathrm{T}^* \to \mathrm{Hom}(\mathrm{F}, \mathrm{N})^* \to \mathrm{Hom}(\mathrm{M}, \mathrm{N})^* \to \mathrm{Ext}^1(\mathrm{T}, \mathrm{R}), \tag{5}$$

$$0 \to \mathrm{Hom}(\mathrm{L}, \mathrm{N})^* \to \mathrm{T}^* \to 0. \tag{6}$$

Putting (5) and (6) together we get the exact sequence.:

$$0 \to \mathrm{Hom}(\mathrm{L}, \mathrm{N})^* \to \mathrm{Hom}(\mathrm{F}, \mathrm{N})^* \to \mathrm{Hom}(\mathrm{M}, \mathrm{N})^* \to \mathrm{Ext}^1(\mathrm{T}, \mathrm{R}). \tag{7}$$

Now, if p is a prime ideal of height at most two, then M_p and N_p are free R_p-modules, as in the beginning of the proof. Hence (1) and (4) imply that T_p is R_p-free, *i.e.* $\mathrm{Ext}^1_R(T, R)_p = (0)$. Denoting by P, the image of the homomorphism $\mathrm{Hom}(M, N)^* \to \mathrm{Ext}^1(T, R)$, then by what precedes, $P_p = (0)$ *i.e.* if

$\mathcal{C}$ = annihilator of P, then height $\mathcal{C} \geqslant 3$.

Since R is Cohen Macaulay, grade P = height $\mathcal{C} \geqslant 3$.

Hence, we have $\mathrm{Ext}^i_R(P, R) = (0)$, for $i = 0, 1, 2$. Breaking (7) into two short exact sequences, as in the case of (2), we obtain

$$0 \to \mathrm{Hom}(L, N)^* \to \mathrm{Hom}(F, N)^* \to U \to 0 \tag{8}$$

$$0 \to U \to \mathrm{Hom}(M, N)^* \to P \to 0 \tag{9}$$

We dualize these sequences and make use of $\mathrm{Ext}^i(P, R) = (0)$ for $i = 0, 1$ in (9). Then putting together the resulting exact sequences we get,

$$0 \to \mathrm{Hom}(M, N)^{**} \to \mathrm{Hom}(F, N)^{**} \to \mathrm{Hom}(L, N)^{**} \to \mathrm{Ext}^1(U, R) \tag{10}$$

From (9) and the facts that $\mathrm{Ext}^1(\mathrm{Hom}(M, N)^*, R) = (0)$ $\mathrm{Ext}^2(P, R) = (0)$, we get $\mathrm{Ext}^1(U, R) = (0)$, *i.e.* the sequence (10) becomes

$$0 \to \mathrm{Hom}(M, N)^{**} \to \mathrm{Hom}(F, N)^{**} \to \mathrm{Hom}(L, N)^{**} \to 0 \tag{11}$$

From (1), applying the functor Ext(, N), we get the exact sequence,

$$\mathrm{Hom}(F, N) \to \mathrm{Hom}(L, N) \to \mathrm{Ext}^1(M, N) \to 0.$$

Then with obvious maps, the diagram

$$\begin{array}{ccccccc} \mathrm{Hom}(F, N) & \to & \mathrm{Hom}(L, N) & \to & \mathrm{Ext}^1(M, N) & \to & 0 \\ \downarrow & & \downarrow & & & & \\ \mathrm{Hom}(F, N)^{**} & \to & \mathrm{Hom}(L, N)^{**} & \to & 0 & & \end{array}$$

is commutative with isomorphic columns. Hence $\mathrm{Ext}^1_R(M, N) = (0)$. This proves our claim.

To conclude, we apply the Lemma [7] after noticing that only the rigidity of N is needed for its validity. We find that the natural homomorphism $M^* \otimes N \to \mathrm{Hom}(M, N)$ is an isomorphism.

THEOREM 2. Suppose R is regular local and M any non-zero R-module. For any non-negative integer n, $\mathrm{P.d.}M \leqslant n$ if and only if the natural homomorphism $\mathrm{Ext}^n_R(M, R) \otimes M \to \mathrm{Ext}^n_R(M, M)$ is surjective, in which case the map is actually bijective.

Remark If $n = 0$, this is a well known criterion for the projectivity of a module.

Proof Suppose $\mathrm{p.d.}M = j$. If $j < n$, clearly $\mathrm{Ext}^n_R(M, R) = (0) = \mathrm{Ext}^n_R(M, M)$. If $j = n$, then it is well known that the natural homomorphism

$\mathrm{Ext}^n_R(M, R) \otimes M \to \mathrm{Ext}^n_R(M, M)$ is an isomorphism [1].

Conversely, suppose the latter homomorphism is surjective, we make use of the fundamental exact sequences of Auslander and Bridger [1]:

$$\mathrm{Tor}^R_2(D\Omega^n M, N) \to \mathrm{Ext}^n_R(M, R) \otimes N \to \mathrm{Ext}^n_R(M, N) \to \mathrm{Tor}^R_1(D\Omega^n M, N) \to 0 \tag{12}$$

$$0 \to \mathrm{Ext}^1_R(D\Omega^n M, N) \to \mathrm{Tor}^R_n(M, N) \to \mathrm{Hom}(\mathrm{Ext}^n_R(M, R), N) \to \mathrm{Ext}^2_R(D\Omega^n M, N) \tag{13}$$

Taking $M = N$ in (12) and using the fact that the homomorphism $\mathrm{Ext}_R^n(M, R) \otimes N \to \mathrm{Ext}_R^n(M, N)$ is surjective, we get,

$\mathrm{Tor}_1^R(D\Omega^n M, M) = (0)$. By the rigidity of M, we get, $\mathrm{Tor}_j^R(D\Omega^n M, M) = (0)$, for $j \geqslant 1$. In particular $\mathrm{Tor}_n^R(D\Omega^n M, M) = (0)$. Then (13), yields $\mathrm{Ext}_R^1(D\Omega^n M, D\Omega^n M) = 0$. Applying [7], we find that $D\Omega^n M$ is projective, *i.e.* $\Omega^n M$ is projective. Hence, p.d.$M \leqslant n$.

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