COMPACT LIE GROUP ACTION
AND EQUIVARIANT BORDISM

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ABSTRACT. Let $G$ be a compact Lie group and $H$ a compact Lie subgroup of $G$ contained in the center of $G$ with $H^m$ the maximal subgroup in the center, $H$ being $H$-boundary. Let $p_r: H^m \to H$ be the projection onto the $r$th factor and $H_r$ be the $r$th factor of $H^m$. Let $\{L_r\}$ be a family of subgroups of $G$ such that $L_r \cap H_r$ is nontrivial. Consider a $G$-manifold $M^n$ with $p_r(G_x \cap H^m)$ trivial or containing $L_r$, for every $x$ in $M^n$. The main result of the paper is that if $\forall x \in M^n$, $p_r(G_x \cap H^m)$ is trivial at least for one $r$, then $M^n$ is a $G$-boundary.

1. Introduction. This paper is a sequel to [3–5]. Conner and Floyd [1] proved that if $\mathbb{Z}_2^k$ acts on a closed manifold $M$ differentiably and without any fixed point, then $M$ is a boundary. Stong [7] showed that if $(M, \theta)$ is a closed $\mathbb{Z}_2^k$-differential manifold without any stationary point, then $(M, \theta)$ is a $\mathbb{Z}_2^k$-boundary. In [3], we extended Stong's result for any finite abelian group of even order by proving the following. Let $G$ be a finite abelian group of even order, $(M, \theta)$ a closed $G$-differential manifold and the elementary 2-group $G_2$ in $G$ acts on $M$ under $\theta$ without any stationary point. Then $(M, \theta)$ is a $G$-boundary. In [4], we initiated this problem for nonabelian groups $S_3$ and dihedral groups. In [5], we have extended the result of [3] for an arbitrary finite group with center of even order. One needs the elementary 2-group $G_2(C)$ of the center of $G$ instead of $G_2$.

In the present note, we consider the action of a compact Lie group and prove that the induced action of the central elementary $H$-subgroup of $G$ determines $G$-bordism. This gives the results of [3–5] in particular cases.\footnote{The author was partially supported by DAE Grant.}

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2. Preliminaries. Let $H$ be a compact Lie group. If there exists an $H$-differential closed manifold $(N, \theta)$ such that the boundary $\partial N = H$ and the restriction of the action $\theta$ to $H$ coincides with the operation in $H$, then we say that the compact Lie group $H$ is $H$-boundary. Let $G$ be a compact Lie group. By the central elementary $H$-group in $G$, we will mean the maximal subgroup $H^n (= H \times \cdots \times H$, $n$ times) contained in the center of $G$.

Consider a compact Lie group $G$ with $H^n$ the central elementary $H$-group in $G$, $H$ being $H$-boundary. Let us fix a point $h_0$ of $H$. Let $p_r: H^n \to H$ denote the projection onto the $r$th factor, $1 \leq r \leq n$. Let $H_r$ denote the subgroup of $H^n$
with \( p_i(H_r) = H \), if \( i = r \) and \( p_i(H_r) = h_0, \) if \( i \neq r \). Consider a family \( \{ L_r \} \) of subgroups \( L_r \) of \( G \) such that \( L_r \cap H_r \) is a nontrivial subgroup of \( H_r, \ 1 \leq r \leq n \).

By an \( \{ L_r \} \)-type action of \( G \), we mean a differential action of \( G \) on a differential manifold such that for every \( x \in M, p_r(G_x \cap H^n) \) is either trivial or contains \( L_r \ \forall r, \ G_x \) being the isotropy group of \( x \). A point \( x \in M \) is said to be a pseudo stationary point if \( p_r(G_x \cap H^n) \) is nontrivial \( \forall r, \ 1 \leq r \leq n \).

A family \( \mathcal{F} \) in \( G \) is a collection of subgroups of \( G \) such that if \( K \in \mathcal{F} \), then all the subgroups of \( K \) and all conjugates of \( K \) are in \( \mathcal{F} \). Let \( \mathcal{F}' \subset \mathcal{F} \) be families in \( G \) such that there exists an \( H \)-boundary subgroup \( H \subset G \) satisfying the following conditions:

(a) no nontrivial subgroup of \( H \) is contained in \( \forall K \in \mathcal{F} \setminus \mathcal{F}' \),
(b) the intersection \( I \) of all the members of \( \mathcal{F} \setminus \mathcal{F}' \) is in \( \mathcal{F} \setminus \mathcal{F}' \),
(c) \( H \) is contained in the center.

We call such a pair \((\mathcal{F}, \mathcal{F}')\) of families an admissible pair in \( G \) with respect to \( H \subset G \).

**Example 2.1.** Consider a family \( \{ L_r \} \) of subgroups of \( G \) such that \( L_r \cap H_r \) is a nontrivial subgroup of \( H_r, \) Let \( \mathcal{F}_r^L \) denote the family of all subgroups \( K \) of \( G \) for which \( p_j(K \cap H^n) \) is trivial at least for one \( j = 1, \ldots, r \) and the nontrivial subgroups \( p_j(K \cap H^n) \) of \( H_j \) contain \( L_j \). Then \( \{ \mathcal{F}_r^L \} \) is an admissible pair of families in \( G \) with respect to \( H_r, 1 \leq r \leq n \), \( \mathcal{F}_0^L \) being the empty family.

**3. \( \mathcal{F}_n^L \)-free action and \( G \)-bordism.** The object of this section is to show that if \( (M, \theta) \) is an \( \mathcal{F}_n^L \)-free closed \( G \)-manifold, then \( (M, \theta) \) is \( G \)-boundary. Let \( \mathfrak{M}_n(G; \mathcal{F}, \mathcal{F}') \) denote the \( (\mathcal{F}, \mathcal{F}') \)-free \( G \)-bordism group for a pair \((\mathcal{F}, \mathcal{F}')\) of families in \( G \). For a given family \( \mathcal{F} \) in \( G \) and a subgroup \( K \subset G \), let \( \mathcal{F}_K \) denote the smallest family in \( G \) containing all the subgroups \([S \cup P], S \in \mathcal{F} \) and \( P \) a subgroup of \( K \).

**Theorem 3.1.** If \((\mathcal{F}, \mathcal{F}')\) is an admissible pair of families in \( G \) with respect to a subgroup \( H \), which is \( H \)-boundary, then the homomorphism

\[
\mathfrak{M}_n(G; \mathcal{F}, \mathcal{F}') \to \mathfrak{M}_n(G; \mathcal{F}_H, \mathcal{F}'_H)
\]

induced by the inclusion map \((\mathcal{F}, \mathcal{F}') \to (\mathcal{F}_H, \mathcal{F}'_H)\) is the zero homomorphism.

**Proof.** Let \([M, \theta]\) be in \( \mathfrak{M}_n(G; \mathcal{F}, \mathcal{F}') \). Let \( F \) denote the fixed points set of \( I \) in \( M \), \( I \) being the intersection of all the members of \( \mathcal{F} \setminus \mathcal{F}' \). Since \( \mathcal{F} \setminus \mathcal{F}' \) is invariant under conjugation, \( I \) is normal in \( G \) so that the action \( \theta \) induces an action on \( F \), which we once again denote by \( \theta \). Let \( \nu \) be the normal bundle of the imbedding of \( F \) in the interior of \( M \) and \( D(\nu) \) be the disc bundle with the action \( \theta^* \) of \( G \) on \( D(\nu) \) induced by the real vector bundle maps covering the action \( \theta \) on \( F \). Since \( F \) is the fixed points set of \( I \), no nontrivial subgroup of \( H \) is contained in \( K \forall K \in \mathcal{F} \setminus \mathcal{F}' \), no point of \( F \) will be fixed by the subgroup \([I \cup P], P \) being a nontrivial subgroup of \( H \), so that \( H \) will act freely on \( F \) and hence on \( D(\nu) \). Let \( F' = F/H \) and \( D'(\nu) = D(\nu)/H \). The actions \( \theta \) and \( \theta^* \) on \( F \) and on \( D(\nu) \) induce actions \( \theta' \) and \( \theta'^* \) on \( F' \) and \( D'(\nu) \) respectively, because \( H \) is contained in the center. Since \( H \) acts freely on \( F \) and \( D(\nu) \), the quotient maps \( \xi_1: F \to F' \) and \( \xi_2: D(\nu) \to D'(\nu) \) are principal \( H \)-bundles. Since \( H \) is \( H \)-boundary, there exists an \( H \)-differential closed manifold \((N, \emptyset)\) such that the boundary \( \partial N = H \) and the restriction of the action \( \theta \) to \( H \) coincides with the operation in \( H \). Consider the fibre bundles \( \xi_1 = \xi_1[N] \) and \( \xi_2 = \xi_2[N] \) associated to the principal \( H \)-bundles.
\(\xi_1\) and \(\xi_2\) respectively. The total space \(E_1 = (F \times N)/H\), where the action of \(H\) on \(F \times N\) is given by \(h(m, t) = (mh, h^{-1}t), h \in H\) and \((m, t) \in F \times N\). Also the boundary \(\partial E_1\) is diffeomorphic to \((F \times H)/H\). Let us take a fixed point \(\tilde{h}\) of \(H\). Define a map \(\eta: (F \times H)/H \to F\) as \(\eta([m, h]) = m\tilde{h}\), where \(h = \tilde{h}\tilde{h}\). Clearly \(\eta\) is a diffeomorphism. Let us define an action \(\psi_1\) of \(G\) on \(E_1\) as \(g[m, t] = [mg, t]\). Then the diffeomorphism \(\eta\) preserves the \(H\)-action. Thus \(E_1\) is a \(G\)-manifold with \(\partial E_1\) being equivariantly diffeomorphic to \(F\). Similarly the total space \(E_2 = \tilde{\xi}_2 = (D(\nu) \times N)/H\), where the action of \(H\) on \(D(\nu) \times N\) is given by \(h(m, t) = (mh, h^{-1}t)\). Consider the action \(\psi_2\) of \(G\) on \(E_2\) as \(g[m, t] = [mg, t]\). Let \(\alpha: E_2 \to E_1\) be the map induced from \(\nu': D(\nu) \to F'\) by going to the fibre bundles; one has the commutative diagram:

\[
\begin{align*}
\xi_2(N): \quad E_2 & \to D(\nu) \\
{\downarrow}_\alpha & \quad {\downarrow}_{\nu'} \\
\xi_1(N): \quad E_1 & \to F'
\end{align*}
\]

Also \(\alpha^{-1}(\partial E_1)\) is diffeomorphic to \(D(\nu)\) and the action \(\psi_2\) on \(\alpha^{-1}(\partial E_1)\) is isomorphic to the action \(\theta^*\) on \(D(\nu)\). Consider

\[
W = (M \times [0, 1]) \cup E_2/\sim
\]

where \(\sim\) is the equivalence relation in \(W\) obtained by identifying \(D(\nu) \times \{1\}\) with \(\alpha^{-1}(\partial E_1)\). Let the action \(\Phi\) of \(G\) on \(W\) be defined by \(\Phi|M \times [0, 1] = \theta \times 1\) and \(\Phi|E_2 = \psi_2\). Take \(V\) to be

\[
(\partial M \times [0, 1]) \cup (M \times \{1\} - (D(\nu) \times \{1\})^c) \cup (\partial E_2 - (\alpha^{-1}(\partial E_1))^c),
\]

where \(^c\) denotes the interior operator. Since \(I\) is the intersection of all members of \(\mathcal{I} - \mathcal{I}'\), \(V\) will be \((\mathcal{I}'_H, \mathcal{I}'_H^c)\)-free. Also \(W\) is \((\mathcal{I}'_H, \mathcal{I}'_H^c)\)-free and \(\partial V\) is diffeomorphic to \(M \cup V\) identifying \(\partial W\) with \(\partial M\). This shows that \([M, \theta]\) is zero in \(\mathfrak{M}_*(G; \mathcal{I}'_H, \mathcal{I}'_H^c)\). \(\Box\)

Let \(\mathcal{A}\) denote the family of all subgroups of \(G\). Following the notations of Example 2.1 and using Theorem 3.1 we get the following

**Corollary 3.2.** For every \(r, 0 \leq r \leq n\), the homomorphism \(\mathfrak{M}_*(G, \mathcal{I}'_{r+1}, \mathcal{I}'_r) \to \mathfrak{M}_*(G; \mathcal{A}, \mathcal{I}'_r)\) induced from the inclusion map \((\mathcal{I}'_{r+1}, \mathcal{I}'_r) \to (\mathcal{A}, \mathcal{I}'_r)\) is the zero one.

**Proof.** Since \((\mathcal{I}'_{r+1}, \mathcal{I}'_r)\) is an admissible pair of families with respect to the subgroup \(H_{r+1}, 0 \leq r \leq n\), and \((\mathcal{I}'_r)_{H_{r+1}} = \mathcal{I}'_r\), Theorem 3.1 gives the corollary. \(\Box\)

**Corollary 3.3.** Let \(M\) be a closed \(G\)-manifold with \(\{L_r\}\)-type of action for some family \(\{L_r\}\) of subgroups of \(G\) such that \(L_r \cap H_r\) is nontrivial. If \(M\) does not have any pseudo stationary point, then \(M\) is a \(G\)-boundary.

**Proof.** It is enough to show that the homomorphism \(\mathfrak{M}_*(G, \mathcal{I}'_n) \to \mathfrak{M}_*(G; \mathcal{A})\) induced from the inclusion map \(\mathcal{I}'_n \to \mathcal{A}\) is the zero one. By Corollary 3.2 and the exact bordism sequence for the triple \((\mathcal{A}, \mathcal{I}'_{r+1}, \mathcal{I}'_r)\), one gets that \(j_*: \mathfrak{M}_*(G; \mathcal{A}, \mathcal{I}'_r) - \mathfrak{M}_*(G; \mathcal{A}, \mathcal{I}'_{r+1})\) is a monomorphism, \(j: (\mathcal{A}, \mathcal{I}'_r) \to (\mathcal{A}, \mathcal{I}'_{r+1})\) is the inclusion map. Therefore the composite

\[
\mathfrak{M}_*(G; \mathcal{A}, \mathcal{I}'_n) \to \mathfrak{M}_*(G; \mathcal{A}, \mathcal{I}'_1) \to \ldots \to \mathfrak{M}_*(G; \mathcal{A}, \mathcal{I}'_n)
\]
is a monomorphism and hence by the bordism exact sequence of the triple $(\mathfrak{X}, \mathfrak{F}_n^L, \mathfrak{F}_0^L)$, one gets that $\mathfrak{M}_*(G; \mathfrak{F}_n^L, \mathfrak{F}_0^L) \to \mathfrak{M}_*(G; \mathfrak{X}, \mathfrak{F}_0^L)$ is the zero homomorphism. This completes the proof, since $\mathfrak{F}_0^L$ is empty. □

REMARK 3.4. Taking $G$ to be a finite abelian group of even order and $H$ to be $\mathbb{Z}_2$, one gets Corollary 3.7 of [3]. Considering $G$ to be a finite group with center of even order and $H$ to be $\mathbb{Z}_2$, one gets Corollary 3.3 of [5].

REMARK 3.5. The case $H = \mathbb{Z}_2$ can also be obtained in a simpler way using the technique used in the Appendix of [2]. The case $G =$ finite group and $H = \mathbb{Z}_2$, has been obtained by Kosniowski [6] using the concept of slices.

REFERENCES